Advanced Topics in Geometry E1 (MTH.B505)

Geodesics

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Exercise 5

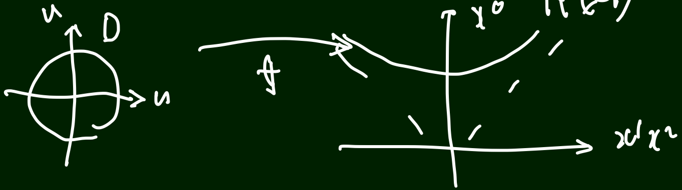
Set the hyperbolic plane

$$\underset{\sim}{H^2(-1)} = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, x^0 > 0\},$$

and take a parametrization $\lambda := 1 - u^2 - v^2$

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}(1+u^2+v^2, 2u, 2v) \in H^2(-1)$$

of $H^2(-1)$, where $D := \{(u,v) \in \mathbb{R}^2 \,;\, u^2+v^2 < 1\}$.



Exercise 5-1

Problem (Ex. 5-1)

Let $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *be an orthonormal frame as*

$$\boldsymbol{e}_0 := \boldsymbol{f}, \qquad \boldsymbol{e}_1 := \frac{\boldsymbol{f}_u}{|\boldsymbol{f}_u|}, \qquad \boldsymbol{e}_2 := \frac{\boldsymbol{f}_v}{|\boldsymbol{f}_v|}.$$

For the induced connection ∇ *of* $H^2(-1)$*,*

- *Compute* $\langle \boxed{\nabla_{\boldsymbol{e}_i}\boldsymbol{e}_j}, \boldsymbol{e}_k\rangle$ *for* i, j *and* k *run over* $\{1,2\}$.
- *Compute* $\nabla_{\boldsymbol{e}_1}\nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2}\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1,\boldsymbol{e}_2]}\boldsymbol{e}_2$.

Exercise 5-1

$\lambda := 1 - u^2 - v^2;$ ✓

$$e_0 = \boldsymbol{f} = \lambda^{-1}\left(1 + u^2 + v^2, 2u, 2v\right)$$

$$\boldsymbol{f}_u = \lambda^{-2}(4u, 2(1 + u^2 - v^2), 4uv),$$

$$\boldsymbol{f}_v = \lambda^{-2}(4v, 4uv, 2(1 - u^2 + v^2), 4uv),$$

$|f_u| = \frac{2}{\lambda}$

$$\begin{cases} \boldsymbol{e}_1 = \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv), \\ \boldsymbol{e}_2 = \lambda^{-1}(2v, 4uv, 1 - u^2 + v^2, 2uv). \end{cases}$$

"check" ∘ $\langle e_1, e_1 \rangle = \lambda^{-2}\left(-4u^2 + (1+u^2-v^2)^2 + 4u^2v^2\right)$

$= \cdots = \lambda^{-2}(1-u^2-v^2)^2 = 1$

$\Rightarrow$

D: the canonical connection on $\mathbb{E}^3_1$

- $D_{\boldsymbol{e}_1}\boldsymbol{e}_1 = \boldsymbol{e}_0 - v\boldsymbol{e}_2$,
- $D_{\boldsymbol{e}_1}\boldsymbol{e}_2 = v\boldsymbol{e}_1$
- $D_{\boldsymbol{e}_2}\boldsymbol{e}_1 = u\boldsymbol{e}_2$
- $D_{\boldsymbol{e}_2}\boldsymbol{e}_2 = \boldsymbol{e}_0 - u\boldsymbol{e}_1$.

$$\nabla_{e_i} e_j = \left[D_{e_i} e_j\right]^T$$

normal

$$\mathbb{E}^3_1 = \underbrace{T_f H^3(-1)}_{\mathrm{Span}\{e_1, e_2\}} \oplus \underbrace{N_f}_{\mathbb{R}e_0}$$

Exercise 5-1

$$e_1 = \frac{\lambda}{2} f_u \qquad \varphi = \varphi(u, v)$$

$$\lambda : = 1 - u^2 - v^2;$$

$$(e_1)\varphi = \frac{\lambda}{2} \varphi_u$$

- $\nabla_{e_1} e_1 = -v e_2$,
- $\nabla_{e_1} e_2 = v e_1$
- $\nabla_{e_2} e_1 = u e_2$
- $\nabla_{e_2} e_2 = -u e_1$.

$$\nabla_{e_1} \underline{\nabla_{e_2} e_2} - \underline{\nabla_{e_2} \nabla_{e_1} e_2} - \underline{\nabla_{[e_1, e_2]} e_2} =$$

$$\nabla_{e_1}(\nabla_{e_2} e_2) = \nabla_{e_1}(-u e_1) = \{e_1(-u)\} e_1 - u \nabla_{e_1} e_1$$

$$= \frac{\lambda}{2}(-u)_u e_1 + uv\, e_2 = \underline{-\frac{\lambda}{2} e_1 + \cancel{uv\, e_2}}$$

$$\nabla_{e_2}(\nabla_{e_1} e_2) = \nabla_{e_2}(v e_1) = \underline{\frac{\lambda}{2} e_1 + \cancel{vu\, e_2}}$$

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = v e_1 - u e_2$$

$$\nabla_{[e_1, e_2]} e_2 = \nabla_{v e_1 - u e_2} e_2 = v \nabla_{e_1} e_2 - u \nabla_{e_2} e_2$$

$$= v^2 e_1 + \underline{u^2 e_1}$$

$$\nabla_{e_1}\nabla_{e_2} e_2 - \nabla_{e_2}\nabla_{e_1} e_2 - \nabla_{[e_1, e_2]} e_2$$

$$= -\lambda e_1 \mp (u^2+v^2) e_1 = (-1 + u^2 + v^2 - u^2 - v^2) e_1$$

$$= -e_1$$

Rem $R(e_1, e_2) e_2$: $\underline{\text{the curvature}}$ tensor of $H^2(-1)$

$2Q$

The Gaussian curvature of $H^2(-1) = -1$

(sectional)

Exercise 5-2

$H^2(-1)$ (alternative parametrization)

Problem (Ex. 5-2)

Let $\widetilde{D} := (0,\infty)\times(-\pi,\pi)$,

$$\widetilde{\boldsymbol{f}} : \widetilde{D} \ni (r,t) \mapsto (\cosh r, \sinh r\cos t, \sinh r \sin t) \in H^2(-1)$$

and set

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{f}}, \qquad \boldsymbol{v}_1 = \widetilde{\boldsymbol{f}}_r, \qquad \boldsymbol{v}_2 = \frac{1}{\sinh r}\widetilde{\boldsymbol{f}}_t.$$

- *Find a parameter change* $\varphi\colon (r,t)\mapsto(u,v) = \bigl(u(r,t), v(r,t)\bigr)$.
- *Find a* 2×2*-matrix valued function* $\Theta = \Theta(r,t)$ *satisfying* $(\boldsymbol{e}_1,\boldsymbol{e}_2) = (\boldsymbol{v}_1,\boldsymbol{v}_2)\Theta$.

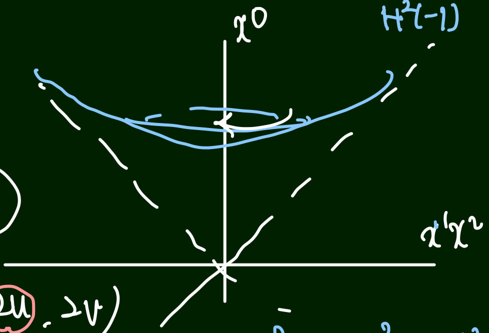
$$\tilde{f}(r,t) = \left(\underline{\cosh r}\ \sinh r \cos t,\ \sinh r \sin t\right)$$

$$= \frac{1}{\underline{1-u^2-v^2}}\left(\underline{1+u^2+v^2},\ 2u,\ 2v\right)$$

$$= f(u,v)$$

$$\cosh r = \frac{1+u^2+v^2}{\underbrace{1-u^2-v^2}_{\lambda}} = \frac{1}{\lambda}(2-\lambda)$$

$$\frac{2}{\lambda} = \cosh r + 1 = 2\cosh^2\frac{r}{2}$$



$$-(x^0)^2 + (x^1)^2 + (x^2)^2 = -1$$

$(x^2 = 0)$

$$-(x^0)^2 + (x^1)^2 = -1$$

$$x^0 > 0$$

$$x^0 = \cosh r$$

$$x^1 = \sinh r$$

$$\frac{2u}{\lambda} = \sinh r \cos t \qquad \frac{2v}{\lambda} = \sinh r \sin t$$

$$\frac{2}{\lambda} = 2\cosh^2 \frac{r}{2}$$

$$u = \frac{\sinh r}{2\cosh^2 \frac{r}{2}} \cos t = \frac{2 \sinh \frac{r}{2} \cosh \frac{r}{2}}{2\cosh^2 \frac{r}{2}} \cos t$$

$$= \tanh \frac{r}{2} \cdot \cos t$$

$$v = \tanh \frac{r}{2} \sin t$$

"polar coordinate" for the hyp. plane

$$\varphi(r, t) = (u, v) = \tanh \frac{r}{2} (\cos t, \sin t)$$

parameter change

Exercise 5-2

$\tilde{f} = (\cosh r, \sinh r \cos t, \sinh r \sin t)$

$\tilde{f}_r$ $\quad |v_1| = 1 \quad$ $\frac{1}{\sinh r}\tilde{f}_t$

orthonormal tangential frame

$$\boldsymbol{v}_1 = (\sinh r, \cosh r\cos t, \cosh r \sin t), \qquad \boldsymbol{v}_2 = (0, -\sin t, \cos t)$$

$$\boldsymbol{e}_1 = \lambda^{-1}(2u, 1+u^2-v^2, 2uv) = (\cos t)\boldsymbol{a} + (0,1,0),$$

$$\boldsymbol{e}_2 = \lambda^{-1}(2u, 1+u^2-v^2, 2uv) = (\sin t)\boldsymbol{a} + (0,0,1)$$

$$\boldsymbol{a} = \left(\sinh r, 2\sinh^2\frac{r}{2}\cos t, 2\sinh^2\frac{r}{2}\sin t\right)$$

$\langle a, v_1\rangle = -2\sinh^2\frac{r}{2}$

- $\langle e_1, v_1\rangle = -2\sinh^2\frac{r}{2}\cos t + \cosh r\cos t = \cos t$
- $\langle e_1, v_2\rangle = -\sin t$

$e_1 = \cos t\, v_1 - \sin t\, v_2$

$e_2 = \sin t\, v_1 + \cos t\, v_2$

$(e_1 \ e_2) = (v_1 \ v_2)\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}$

$\in SO(2)$ orthogonal

Gauge transf. of frames

Exercise 5-2

$$\boldsymbol{v}_1 = (\sinh r, \cosh r \cos t, \cosh r \sin t), \qquad \boldsymbol{v}_2 = (0, -\sin t, \cos t)$$

$$\boldsymbol{e}_1 = \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv) = (\cos t)\boldsymbol{a} + (0, 1, 0),$$

$$\boldsymbol{e}_2 = \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv) = (\sin t)\boldsymbol{a} + (0, 0, 1)$$

$$\boldsymbol{a} = \left(\sinh r, 2\sinh^2 \frac{r}{2} \cos t, 2\sinh^2 \frac{r}{2} \sin t\right)$$

(Gauge transf of the connections) 2Q

@ compute $\nabla_{v_i} v_j$

find a relationship between

$(\nabla_{e_i} e_j)$ & $(\nabla_{v_i} v_j)$