Advanced Topics in Geometry E1 (MTH.B505)

Geodesics

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2023/05/30

Exercise 5

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0\},$$

and take a parametrization

$$\boldsymbol{f} : D \ni (u, v) \mapsto \frac{1}{1 - u^2 - v^2}\bigl(1 + u^2 + v^2, 2u, 2v\bigr) \in H^2(-1)$$

of $H^2(-1)$, where $D := \{(u, v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\}$.

Exercise 5-1

Problem (Ex. 5-1)

Let $[\boldsymbol{e}_0(u,v), \boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ *be an orthonormal frame as*

$$\boldsymbol{e}_0 := \boldsymbol{f}, \qquad \boldsymbol{e}_1 := \frac{\boldsymbol{f}_u}{|\boldsymbol{f}_u|}, \qquad \boldsymbol{e}_2 := \frac{\boldsymbol{f}_v}{|\boldsymbol{f}_v|}.$$

For the induced connection ∇ *of* $H^2(-1)$*,*

- *Compute* $\langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$ *for* i*,* j *and* k *run over* $\{1, 2\}$*.*
- *Compute* $\nabla_{\boldsymbol{e}_1} \nabla_{\boldsymbol{e}_2} \boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2} \nabla_{\boldsymbol{e}_1} \boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1, \boldsymbol{e}_2]} \boldsymbol{e}_2$*.*

Exercise 5-1

$\lambda := 1 - u^2 - v^2$;

$$
\begin{aligned}
\boldsymbol{f} &= \lambda^{-1}\bigl(1 + u^2 + v^2, 2u, 2v\bigr) \\
\boldsymbol{f}_u &= \lambda^{-2}(4u, 2(1 + u^2 - v^2), 4uv), \\
\boldsymbol{f}_v &= \lambda^{-2}(4v, 4uv, 2(1 - u^2 + v^2), 4uv), \\
\boldsymbol{e}_1 &= \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv), \\
\boldsymbol{e}_2 &= \lambda^{-1}(2v, 4uv, 1 - u^2 + v^2, 2uv).
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
D_{\boldsymbol{e}_1}\boldsymbol{e}_1 &= \boldsymbol{e}_0 - v\boldsymbol{e}_2, \qquad & D_{\boldsymbol{e}_1}\boldsymbol{e}_2 &= v\boldsymbol{e}_1 \\
D_{\boldsymbol{e}_2}\boldsymbol{e}_1 &= u\boldsymbol{e}_2 & D_{\boldsymbol{e}_2}\boldsymbol{e}_2 &= \boldsymbol{e}_0 - u\boldsymbol{e}_1.
\end{aligned}
$$

Exercise 5-1

$\lambda \colon = 1 - u^2 - v^2;$

$$\begin{aligned} \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_1 &= -v\boldsymbol{e}_2, & \nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 &= v\boldsymbol{e}_1 \\ \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_1 &= u\boldsymbol{e}_2 & \nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 &= -u\boldsymbol{e}_1. \end{aligned}$$

$\nabla_{\boldsymbol{e}_1}\nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2}\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1,\boldsymbol{e}_2]}\boldsymbol{e}_2 =$

Exercise 5-2

Problem (Ex. 5-2)

Let $\widetilde{D} := (0, \infty) \times (-\pi, \pi)$*,*

$$\widetilde{\boldsymbol{f}} : \widetilde{D} \ni (r, t) \mapsto (\cosh r, \sinh r \cos t, \sinh r \sin t) \in H^2(-1)$$

and set

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{f}}, \qquad \boldsymbol{v}_1 = \widetilde{\boldsymbol{f}}_r, \qquad \boldsymbol{v}_2 = \frac{1}{\sinh r}\widetilde{\boldsymbol{f}}_t.$$

- *Find a parameter change* $\varphi\colon (r, t) \mapsto (u, v) = \bigl(u(r, t), v(r, t)\bigr)$.
- *Find a* 2×2*-matrix valued function* $\Theta = \Theta(r, t)$ *satisfying* $(\boldsymbol{e}_1, \boldsymbol{e}_2) = (\boldsymbol{v}_1, \boldsymbol{v}_2)\Theta$.

Exercise 5-2

$$
\begin{aligned}
\boldsymbol{v}_1 &= (\sinh r, \cosh r \cos t, \cosh r \sin t), \qquad \boldsymbol{v}_2 = (0, -\sin t, \cos t) \\
\boldsymbol{e}_1 &= \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv) = (\cos t)\boldsymbol{a} + (0, 1, 0), \\
\boldsymbol{e}_2 &= \lambda^{-1}(2u, 1 + u^2 - v^2, 2uv) = (\sin t)\boldsymbol{a} + (0, 0, 1) \\
\boldsymbol{a} &= \left(\sinh r, 2\sinh^2 \frac{r}{2} \cos t, 2\sinh^2 \frac{r}{2} \sin t\right)
\end{aligned}
$$