Advanced Topics in Geometry E1 (MTH.B505)

Geodesics

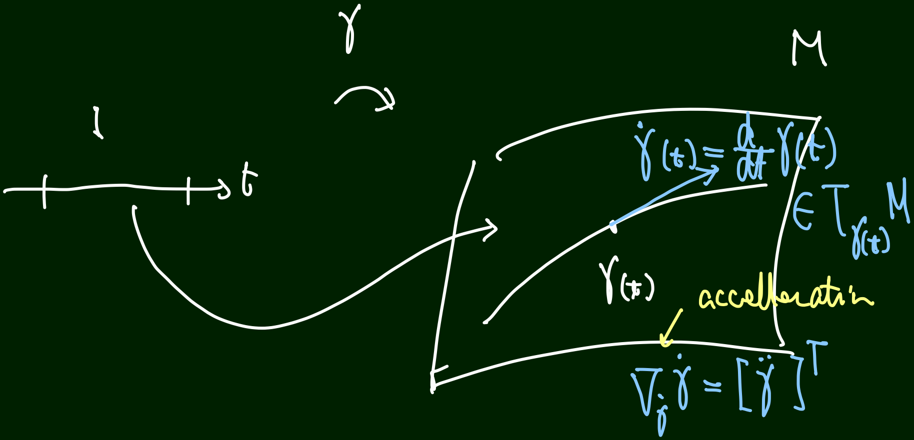
Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-e1/

Tokyo Institute of Technology

2023/05/30

Set up

- (M, g): a (pseudo) Riemannian manifold
- ∇: the Levi-Civita connection (determined uniquely from g)



Pregeodesics and Geodesics

Geodesic (測地線)
Pregeodesic (準測地線)

Definition

A curve $\gamma = \gamma(t)$ is called a pregeodesic if $\nabla_{\dot\gamma}\dot\gamma$ is parallel to $\dot\gamma$, that is, there exists a smooth function $\varphi(t)$ in t such that $\boxed{\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma.}$

$\nabla_{\dot\gamma}\dot\gamma \parallel \dot\gamma$

Definition

A curve γ is called a geodesic if $\boxed{\nabla_{\dot\gamma}\dot\gamma = \mathbf{0}}$ holds identically.

Geodesic $\Rightarrow$ Pregeod.

Lemma

If γ is a geodesic, then $\langle\dot\gamma, \dot\gamma\rangle$ is constant. (in t)

$$\frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle = \langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle + \langle\dot\gamma, \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{0}\rangle$$

directional derivative w.r.to $\dot\gamma$

$$= 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0$$

- a geodesic is constant speed

Pregeodesics and Geodesics

Definition

A curve $\gamma = \gamma(t)$ is called a pregeodesic if $\nabla_{\dot\gamma}\dot\gamma$ is parallel to $\dot\gamma$, that is, there exists a smooth function $\varphi(t)$ in t such that $\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma$.

Definition

A curve γ is called a geodesic if $\nabla_{\dot\gamma}\dot\gamma = \mathbf{0}$ holds identically.

Lemma

If γ is a geodesic, then $\langle\dot\gamma, \dot\gamma\rangle$ is constant.

Rem Geodesic : does depend on parameter change

$$\begin{cases} \gamma(t) & t = t(s) : \text{a parameter change} \\ \tilde\gamma(s) = \gamma(t(s)) & \end{cases}$$

$$\tilde{\gamma}(s) = \gamma(t(s))$$

$$\frac{d\tilde{\gamma}}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt} \; ; \quad \nabla_{\frac{d\tilde{\gamma}}{ds}} \frac{d\tilde{\gamma}}{ds} = \nabla_{\frac{d\tilde{\gamma}}{ds}} \frac{dt}{ds}\frac{d\gamma}{dt}$$

$$= \frac{d^2t}{ds^2}\frac{d\gamma}{dt} + \frac{dt}{ds}\nabla_{\frac{dt}{ds}\frac{d\gamma}{dt}}\frac{d\gamma}{dt}$$

$$= \underbrace{\frac{d^2t}{ds^2}}_{\neq 0}\frac{d\gamma}{dt} + \left(\frac{dt}{ds}\right)^2 \nabla_{\dot{\gamma}}\dot{\gamma}$$

- Pregeodesic: does not depend on choice of parameters.

Lemma

Let $\gamma\colon I \ni t \mapsto \gamma(t) \in M$ be a pre geodesic, where $I \subset \mathbb{R}$ is an interval. Then there exists a parameter change $t = t(s)$ such that $\tilde{\gamma}(s) = \gamma(t(s))$ is a geodesic.

$$\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma \qquad s := s(t) = \int_{t_0}^{t}\left(\exp\int_{t_0}^{u}\varphi(\tau)d\tau\right)du$$

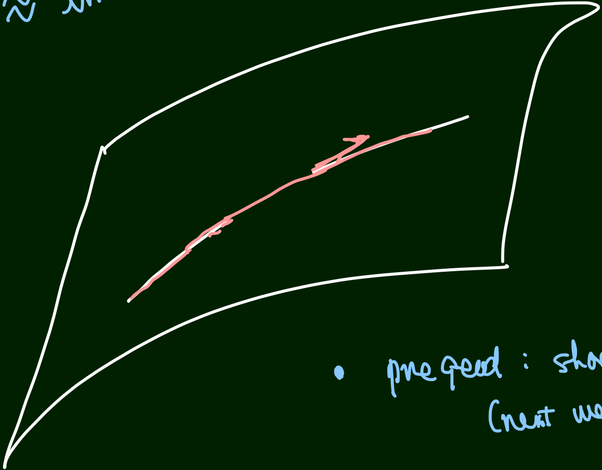
$$\frac{ds}{dt} > 0 \Rightarrow t = t(s) \qquad t_0 \in I$$

$\tilde\gamma(s) = \gamma(t(s))$: the desired curve

$$\left(\nabla_{\tilde\gamma'}\tilde\gamma' = 0\right)$$

pre geodesic

≈ line (inertial motion)



- pre geod : shortest path (next week)

Existence and Uniqueness

$\nabla_{\dot\gamma}\dot\gamma = 0$ $\quad \gamma(t) = (x^1(t) \cdots x^n(t))$

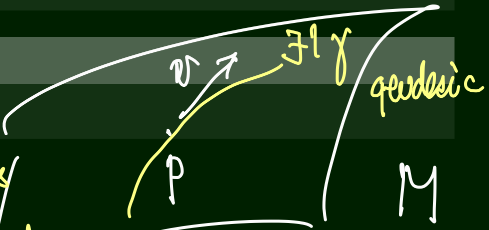
ordinary diff. eq of second order

Fact

For each $p \in M$ and $\boldsymbol{v} \in T_pM$, there exists unique geodesic $\gamma_{p,\boldsymbol{v}} : I \to M$, where I is an interval including 0 such that $\gamma(0) = p$ and $\dot\gamma(0) = \boldsymbol{v}$.

Proposition

$\gamma_{p,k\boldsymbol{v}}(t) = \gamma_{p,\boldsymbol{v}}(kt)$.



- Both sides are geodesics
- Both sides are p at $t=0$
- $\frac{d}{dt}\gamma_{p,kv}(0) = kv$. $\quad \frac{d}{dt}\gamma_{p,v}(kt)\Big|_{u=0} = kv$

Existence and Uniqueness

$\mathbb{E}^n : \nabla_{\dot\gamma}\dot\gamma = \ddot\gamma$

$\ddot\gamma(t) = 0$

$\gamma(t)$

$= tv + a$

: line

Example

$$S^n := \{\boldsymbol{x} \in \mathbb{E}^{n+1} ; \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

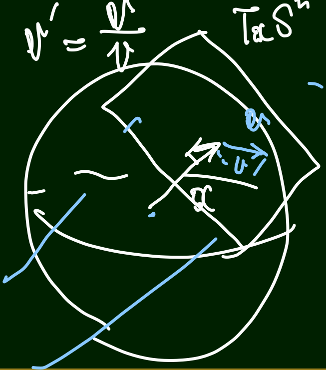
$x \in S^n \quad T_x S^n = x^\perp$

$v \in T_x S^n \quad |v| =: v \quad v' = \frac{v}{v}$

$\gamma(t) := (\cos vt)\, x + (\sin vt)\, v'$

① $\langle \gamma, \gamma \rangle = 1$ $\begin{cases} \langle x, x \rangle = 1 \\ \langle x, v' \rangle = 0 \\ \langle v', v' \rangle = 1 \end{cases}$

i.e. γ is a curve on S^n



$$\underline{\gamma(t) = (\cos vt)\, x + (\sin vt)\, v'} \qquad v' = \frac{v}{v}$$

$$v = |v|$$

- $\gamma : \mathbb{R} \longrightarrow S^n$
- $\ddot{\gamma} = -v^2 \gamma(t) \perp T_{\gamma(t)} S^n \qquad [\ddot{\gamma}]^T = \nabla_{\dot{\gamma}} \dot{\gamma} = 0$

γ is a geodesic

- $\gamma(0) = x, \quad \dot{\gamma}(0) = v \cdot v' = v \cdot \frac{v}{v} = v$

i.e. $\gamma(t) = \gamma_{x,v}(t)$

great circle

$\gamma(\mathbb{R}) = S^n \cap \operatorname{Span}\{x, v\}$

Completeness

完備性

Definition

A pseudo Riemannian manifold (M, g) is said to be complete if all geodesics are defined on whole on $\mathbb{R}$.

- $\gamma(t) = \gamma_{p,v}(t)$ $t \in \mathbb{R}$ for $\forall p, \forall v$

Example $\mathbb{E}^{n+r}_r$: $\gamma_{x,v}(t) = x + tv$ complete

S^n : $\gamma_{x,v}(t) = (\cos vt)\, x + (\sin vt) \dfrac{v}{v}$

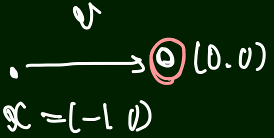
$v = |v|$

$\mathbb{E}^2 \setminus \{(0,0)\}$

$\mathbb{E}^2$

$x = (-1 \;\; 0)$

$v = (1. \;\; 0)$



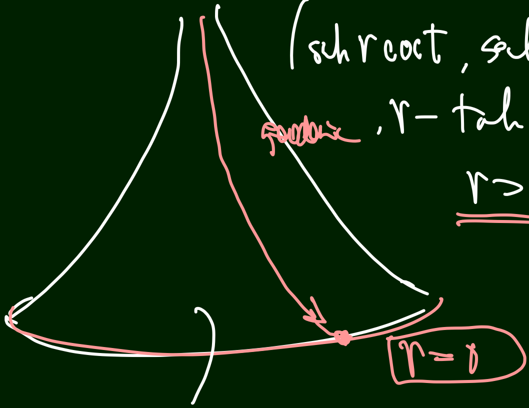
$$\gamma_{x,v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} t-1 \\ 0 \end{pmatrix} \qquad -\infty < t < 1 \qquad \text{incomplete}$$

The pseudosphere

Beltrami's

incomplete



$(\operatorname{sech} r \cos t, \operatorname{sech} r \sin t, r - \tanh r)$

$r > 0$

$K = -1$

(Hilbert's theorem

$\nexists$ complete surface with $K = -1$ in $\mathbb{E}^3$)

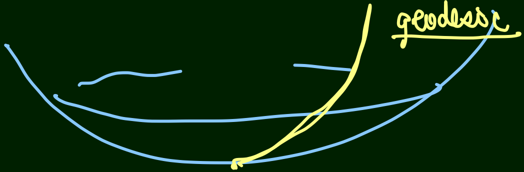
Exercise 6-1

Problem (Ex. 6-1)

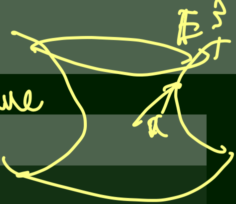
Let

$$\boldsymbol{f}(r,t) := (\cosh r, \sinh r \cos t, \sinh r \sin t)^T \in H^2(-1)$$

be a parametrization in $H^2(-1)$. *Show that* $\gamma(r)\colon r \mapsto \boldsymbol{f}(r,t) \in H^2(-1)$ *is a geodesic for each fixed value* t.



Exercise 6-2



de Sitter plane

Problem (Ex. 6-2)

$S^2_1 := \{\boldsymbol{x} \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$
$T_{\boldsymbol{x}} S^3_1 = \boldsymbol{x}^{\perp}$
$\underline{\boldsymbol{x} \in S^2_1}$, $\underline{\boldsymbol{v} \in T_{\boldsymbol{x}} S^2_1}$

causality

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \textit{if } \langle \boldsymbol{v}, \boldsymbol{v}\rangle < 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \textit{if } \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0, \\ (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \textit{if } \langle \boldsymbol{v}, \boldsymbol{v}\rangle > 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v}\rangle|^{1/2}$ *and* $\boldsymbol{v}' := \boldsymbol{v}/v$. *Show that* $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ *is a geodesic on* S^3_1 *with* $\gamma(0) = \boldsymbol{x}$ *and* $\dot\gamma(0) = \boldsymbol{v}$.

S^2_1

complete Lorentzian manifold.