Advanced Topics in Geometry E1 (MTH.B505)

Geodesics

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-e1/

Tokyo Institute of Technology

2023/05/30

Set up

- (M, g): a pseudo Riemannian manifold
- ∇: the Levi-Civita connection

Pregeodesics and Geodesics

Definition

A curve $\gamma = \gamma(t)$ is called a <u>pregeodesic</u> if $\nabla_{\dot{\gamma}}\dot{\gamma}$ is parallel to $\dot{\gamma}$, that is, there exists a smooth function $\varphi(t)$ in t such that $\nabla_{\dot{\gamma}}\dot{\gamma} = \varphi\dot{\gamma}$.

Definition

A curve γ is called a <u>geodesic</u> if $\nabla_{\dot{\gamma}}\dot{\gamma} = \mathbf{0}$ holds identically.

Lemma

If γ is a geodesic, then $\langle\dot{\gamma}, \dot{\gamma}\rangle$ is constant.

Pregeodesics and Geodesics

Definition

A curve $\gamma = \gamma(t)$ is called a <u>pregeodesic</u> if $\nabla_{\dot\gamma}\dot\gamma$ is parallel to $\dot\gamma$, that is, there exists a smooth function $\varphi(t)$ in t such that $\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma$.

Definition

A curve γ is called a <u>geodesic</u> if $\nabla_{\dot\gamma}\dot\gamma = \mathbf{0}$ holds identically.

Lemma

If γ is a geodesic, then $\langle\dot\gamma, \dot\gamma\rangle$ is constant.

Pregeodesics and Geodesics

Lemma

Let $\gamma\colon I \ni t \mapsto \gamma(t) \in M$ *be a geodesic, where* $I \subset \mathbb{R}$ *is an interval. Then there exists a parameter change* $t = t(s)$ *such that* $\tilde{\gamma}(s) = \gamma(t(s))$ *is a geodesic.*

Existence and Uniqueness

Fact

For each $p \in M$ *and* $\boldsymbol{v} \in T_pM$*, there exists unique geodesic* $\gamma_{p,\boldsymbol{v}} \colon I \to M$*, where* I *is an interval including* 0 *such that* $\gamma(0) = p$ *and* $\dot{\gamma}(0) = \boldsymbol{v}$*.*

Proposition

$\gamma_{p,k\boldsymbol{v}}(t) = \gamma_{p,\boldsymbol{v}}(kt)$.

Existence and Uniqueness

Example

$$S^n := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$$

Completeness

Definition

A pseudo Riemannian manifold (M, g) is said to be complete if all geodesics are defined on whole on $\mathbb{R}$.

Exercise 6-1

Problem (Ex. 6-1)

Let

$$\boldsymbol{f}(r,t) := (\cosh r, \sinh r \cos t, \sinh r \sin t)^T \in H^2(-1)$$

be a parametrization in $H^2(-1)$*. Show that* $\gamma(r)\colon r \mapsto \boldsymbol{f}(r,t) \in H^2(-1)$ *is a geodesic for each fixed value* t*.*

Exercise 6-2

Problem (Ex. 6-2)

$S^2_1 := \{\boldsymbol{x} \in \mathbb{E}^3_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$
$T_{\boldsymbol{x}} S^3_1 = \boldsymbol{x}^\perp$
$\boldsymbol{x} \in S^2_1$, $\boldsymbol{v} \in T_{\boldsymbol{x}} S^2_1$

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0, \\ (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v} \rangle|^{1/2}$ *and* $\boldsymbol{v'} := \boldsymbol{v}/v$. *Show that* $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ *is a geodesic on* S^3_1 *with* $\gamma(0) = \boldsymbol{x}$ *and* $\dot\gamma(0) = \boldsymbol{v}$.