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Info. Sheet 6; Advanced Topics in Geometry E1 (MTH.B505)

Informations

- Next week (June 06) is the final class of MTH.B505.
- Please fill the form "Course Survay" in T2SCHOLA.

Corrections

- Lecture Note, page 17, line 2: We let $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ the set of ... ⇒ We let $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ be the set of ...
- Lecture Note, page 17, equation (5.1): $\langle Z, [X, Y]\rangle \Rightarrow \langle Z, [Y, X]\rangle$
- Lecture Note, page 18, line 2 of Example 5.6: $(r, n) \Rightarrow (n, r)$
- Lecture Note, page 18, line 3 of Lemma 5.7: $\langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0 \textit{for all} \Rightarrow \langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0$ *for all*
- Lecture Note, page 18, line −17: $\dim \mathbb{E}^{n+r} \Rightarrow \dim \mathbb{E}^{n+r}_r$
- Lecture Note, page 18, line 2 of Theorem 5.9: *We set for X, Y* ⇒ *We set $\nabla_X Y$ for X, Y*
- Lecture Note, page 18, line −3: yields ⇒ yield
- Lecture Note, page 19, line 2 of Problem 5-1: $\boldsymbol{f}_{\boldsymbol{u}} \Rightarrow \boldsymbol{f}_u$, $\boldsymbol{f}_{\boldsymbol{v}} \Rightarrow \boldsymbol{f}_v$.

Students' comments

- 5-1 で $\nabla_X Y = [D_X Y]^{\mathrm{T}}$ を使ってここまでかんたんに計算できるのは良いなと思った.

  Lecturer's comment 部分多様体と思うと計算がしやすいことはしばしばありますね.

- スライドのペンを可能であれば少し細くしていただきたいです．u, v, 0 あたりの文字がつぶれて同じに見えることがあります.

  Lecturer's comment 了解.

Q and A

Q 1: $M \subset \mathbb{E}^{n+r}_r$ の接束に入る接続のうち，Levi-Civita 以外で有名な接続はあるのでしょうか？
Connections on submanifold of a (pseudo) Euclidean space different from the Levi-Civita.

A: For a submanifold $M \subset \mathbb{E}^{n+r}$, consider a (not necessarily orthogonal) direct sum decomposition $\mathbb{E}^{n+r} = T_pM \oplus N_p$. Then $\nabla_X Y := [D_X Y]^{\mathrm{T}}$ is a linear connection on M, with respect to the decomposition. For example, classical affine hypersurface theory, in Blaschke's manner, the special transversal vector field ξ along a hypersurface $M \subset \mathbb{E}^{n+1}$ and induced connection ∇ is considered. In the context of (classical) information geometry, important geometric structure is a family of connections, e-connection, m-connection and α-connections for $\alpha \in (-1, 1)$.

Q 2: Lie bracket をある種のベクトル場による微分，Riemannian connection もある種の説ベクトルによる微分ということで 2 つにつながりがあるというのはわかるのですが, $\nabla_X Y - \nabla_Y X = [X, Y]$ にいつまでもなじむことができません．なにか式だけでなく解釈がないのか知りたいです.
Interpretation of $\nabla_X Y - \nabla_Y X = [X, Y]$

A: For a linear connection ∇, $T(X,Y) \colon = \nabla_X Y - \nabla_Y X - [X,Y]$ is a tensor field, called the **torsion tensor**. In fact, $T(fX,Y) = T(X,fY) = fT(X,Y)$ holds, which implies tensority of T. One of the condition to be the Riemannian connection asserts that T vanishes identically, in other words, **torsion free**. The connection obtained by a way as in the previous answer is automatically torsion free. So vanishing torsion is a necessary condition that the manifold can be realized as a submanifold, so called an **inexorability condition**. Moreover, the Ricci tensor induced from ∇ is symmetric 2-tensor if ∇ is torsion free. Connections with non-vanishing torsion recently is treated, for example, a context of quantum information geometry.