(Geodesics. Completeness)

Advanced Topics in Geometry E1 (MTH.B505)

Hopf-Rinow's theorem

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Exercise 6-1

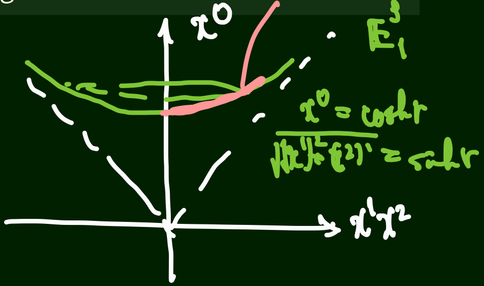
Problem (Ex. 6-1)

Let

$$\boldsymbol{f}(r,t) := (\cosh r, \sinh r \cos t, \sinh r \sin t)^T \in H^2(-1)$$

be a parametrization in $H^2(-1)$. *Show that* geodesic
$\gamma(r) \colon r \mapsto \boldsymbol{f}(r,t) \in H^2(-1)$ *is a geodesic for each fixed value* t.

fix t



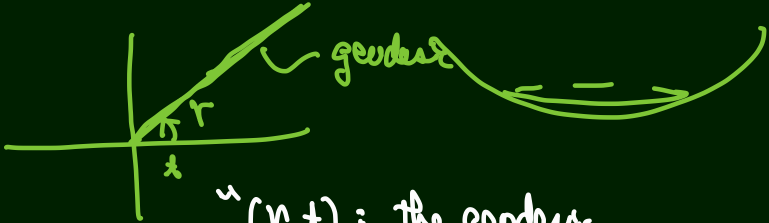
$$\gamma(r) = (\cosh r,\ \sinh r \cos t,\ \sinh r \sin t)$$

$$\underbrace{\ddot{\gamma}(r) = \gamma(r)}$$

$$\gamma(r)^{\perp}\ T_{\gamma(r)}H^2(-1)$$

☺ $\cosh r$ & $\sinh r$ are solutions of ord. diff. eq $\ddot{x} = x$ | $\cos r$, $\sin r$ $\ddot{x} = -x$

$$[\ddot{\gamma}(r)]^T = \nabla_{\dot{\gamma}}\dot{\gamma} = 0 \Rightarrow \text{geodesic.}$$



"(r, t) : the geodesic polar coordinate system"

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cf. the spherical case

Exercise 6-2

de Sitter plane

Problem (Ex. 6-2)

$\langle\,,\,\rangle|_{T_x S^2_1}$ sign $(+,-)$

Lorentzian

$S^2_1 := \{\boldsymbol{x} \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$
$T_{\boldsymbol{x}} S^2_1 = \boldsymbol{x}^\perp$
$\boldsymbol{x} \in S^2_1$, $\boldsymbol{v} \in T_{\boldsymbol{x}} S^2_1$

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0, \\ (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v} \rangle|^{1/2}$ and $\boldsymbol{v}' := \boldsymbol{v}/v$. Show that $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ is a geodesic on S^2_1 with $\gamma(0) = \boldsymbol{x}$ and $\dot\gamma(0) = \boldsymbol{v}$.

$\gamma_{x,v}(t)$ is defined on whole $\mathbb{R}$

S^2_1 is complete Lorentzian manifold.

$x \in S^2_1 \quad (\langle x, x\rangle = 1)$

$v \in T_x S^2_1 \quad (\langle x, v\rangle = 0)$

- $\langle v, v\rangle > 0$ (for simplicity, assume $\langle v, v\rangle = 1$)

$$\gamma(t) = \cos t\, x + \sin t\, v$$

$$\langle \gamma, \gamma\rangle = \cos^2 t\, \underset{1}{\langle x, x\rangle} + 2\cos t \sin t\, \overset{0}{\langle x, v\rangle} + \sin^2 t\, \underset{1}{\langle v, v\rangle}$$

$$= \cos^2 t + \sin^2 t = 1$$

i.e. $\gamma : \mathbb{R} \longrightarrow S^2_1$

$\ddot{\gamma} = -\gamma \perp T_\gamma S^2_1$ i.e. $[\ddot{\gamma}]^T = 0$

• $\langle v, v\rangle < 0$ $\boxed{\langle v, v\rangle = -1}$

$$\gamma(t) = \cosh t\, x + \sinh t\, v$$

$$\langle \gamma, \gamma\rangle = \cosh^2 t \overset{1}{\langle x, x\rangle} + 2\cosh t \sinh t \overset{0}{\langle x, v\rangle} + \sinh^2 t \underbrace{\langle v, v\rangle}_{-1}$$

$$= \cosh^2 t - \sinh^2 t$$

$$= 1$$

i.e. $\gamma : \mathbb{R} \to S^3_1$

$\ddot{\gamma} = \gamma \Rightarrow \gamma$ is a geodesic

• $\langle v, v \rangle = 0 \qquad (v \neq 0)$

$$\gamma(t) = x + tv$$

$$\langle \gamma, \gamma \rangle = \overset{1}{\langle x, x \rangle} + t \overset{0}{\langle x, v \rangle} + t^2 \overset{0}{\underline{\langle v, v \rangle}}$$

$$= \langle x, x \rangle = 1$$

$$\gamma : \mathbb{R} \longrightarrow S^2_1$$

$\ddot{\gamma} = 0 \qquad$ i.e. γ is a geodesic

In general : (M, g) : a Lorentzian manifold

$T_pM \ni v$ is

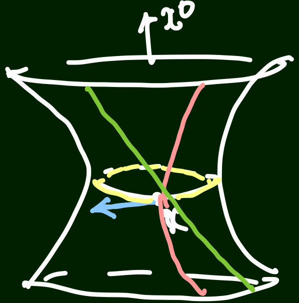
spacelike if $\langle v, v \rangle > 0$

timelike if $\langle v, v \rangle < 0$

lightlike if $\langle v, v \rangle = 0$ $(v \neq 0)$

terminology of Relativity.

geodesics $\begin{cases} \text{timelike geodesic} \\ \text{light like geodesic} \\ \text{spacelike geodesic} \end{cases}$



S^2_1

(ruled surface)

$\rightarrow x^1, x^2$

$x = (0, 1, 0)$

$v = (0, 0, 1) \in T_x S^2_1 \quad \gamma_{x,v} = (0, \cos t, \sin t)$

spacelike

$v = (1, 0, 0) \quad \gamma_{x,v} = (\sinh t, \cosh t, 0)$

timelike

$\langle v, v \rangle = 0$

$v = (1, 0, 1) \quad \gamma_{x,v} = (t, 1, t)$

lightlike