Advanced Topics in Geometry E1 (MTH.B505)

Hopf-Rinow's theorem

Kotaro Yamada

kotaro@math.titech.ac.jp

http://www.math.titech.ac.jp/~kotaro/class/2023/geom-e1/

Tokyo Institute of Technology

2023/06/06

Exercise 6-1

Problem (Ex. 6-1)

Let

$$\boldsymbol{f}(r,t) := (\cosh r, \sinh r \cos t, \sinh r \sin t)^T \in H^2(-1)$$

be a parametrization in $H^2(-1)$. *Show that* $\gamma(r)\colon r \mapsto \boldsymbol{f}(r,t) \in H^2(-1)$ *is a geodesic for each fixed value* t.

Exercise 6-2

Problem (Ex. 6-2)

$S^2_1 := \{\boldsymbol{x} \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$
$T_{\boldsymbol{x}} S^2_1 = \boldsymbol{x}^\perp$
$\boldsymbol{x} \in S^2_1$, $\boldsymbol{v} \in T_{\boldsymbol{x}} S^2_1$

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0, \\ (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \text{if } \langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0, \end{cases}$$

where $v := |\,\langle \boldsymbol{v}, \boldsymbol{v} \rangle\,|^{1/2}$ *and* $\boldsymbol{v}' := \boldsymbol{v}/v$. *Show that* $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ *is a geodesic on* S^2_1 *with* $\gamma(0) = \boldsymbol{x}$ *and* $\dot{\gamma}(0) = \boldsymbol{v}$.