Advanced Topics in Geometry E1 (MTH.B505)

Hopf-Rinow's theorem

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Set up

(M, g): a connected Riemannian manifold

$\Updownarrow$

arcwise - connected

pseudo Riem mfd,

the Hopf-Rinow-Type properties does not hold.

Thm connected & locally arcwise connected
$\Rightarrow$ arcwise connected.

Length

Definition

Let $\gamma\colon [a,b] \to M$ be a piecewize C^1-curve, where $[a,b]$ is a closed interval on $\mathbb{R}$. The integral

$$\mathcal{L}(\gamma) := \int_a^b |\dot{\gamma}(t)|\, dt$$

is called the length of γ.

$\left(\frac{1}{\lambda} g_0 < g < \lambda g_0 \right.$ locally

Euclidean metric $\left. \right)$

Distance

p ≠ q ⇒ d(p, q) > 0

Lemma

For points p, $q \in M$, set

$$d(p,q) := \inf\{\mathcal{L}(\gamma)\,;\, \gamma \in \mathcal{C}_{p,q}\}\colon M \times M \to \mathbb{R},$$

$$\left(\mathcal{C}_{p,q} := \left\{\gamma\colon [a,b] \to M\,;\, \begin{array}{l}\gamma \textit{ is a piecewize } C^1\textit{-curve} \\ \textit{with } \gamma(a) = p \textit{ and } \gamma(b) = q\end{array}\right\}\right)$$

is a distance function on M, that is, it satisfies the axiom nontrivial

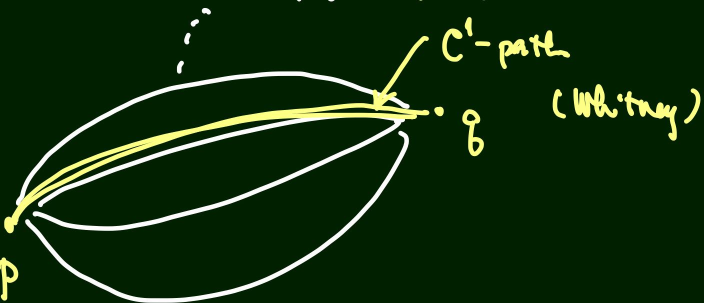
- $d(p,q) \geqq 0$ *for any p, $q \in M$. The equality holds iff $p = q$.*
- $d(p,q) = d(q,p)$.
- $d(p,q) + d(q,r) \geqq d(q,r)$

of distance (compatible to the topology of M).

direct conclusion of the definition

d: the Riemannian distance with respect to g.

$(\mathcal{C}_{p,q} \neq \emptyset)$ $(M$



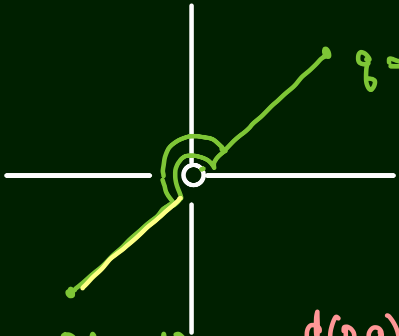
$\mathcal{L} : \mathcal{C}_{p,q} \ni \gamma \longmapsto \mathcal{L}(\gamma) \in \mathbb{R}$ bounded from below.

$\exists \inf \mathcal{L}(\mathcal{C}_{p,q}) =: d(p,q)$
$\in [0,\infty)$

- "Riem. distance of p, q" is the length of the shortest path joining p & q.

not true in general.

$M = \mathbb{E}^2 \setminus \{0\}$



$\forall \varepsilon > 0$

$\exists$ path γ with

$2\sqrt{2} \leq L(\gamma) \leq 2\sqrt{2} + \varepsilon$

$\underline{d(p,q) = 2\sqrt{2}}$

$\nexists$ shortest path

The shortest geodesic

Proposition Assume $\exists$ shortest path

For p, $q \in M$, a curve $\gamma \in \mathcal{C}_{p,q}$ satisfying $d(p,q) = \mathcal{L}(\gamma)$ is a pregeodesic.

the shortest path $\Rightarrow$ pregeodesic.

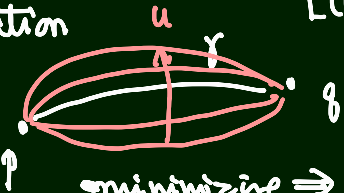
Definition

The geodesic $\gamma \in \mathcal{C}_{p,q}$ satisfying $\mathcal{L}(\gamma) = d(p,q)$ is called the minimizing geodesic or the shortest geodesic joining p and q.

(proof) " variation "

$L(u) = \mathcal{L}(\gamma_u(t))$

$\gamma_0 = \gamma$



minimizing $\Rightarrow \left.\frac{dL}{du}\right|_u = 0$

$\Rightarrow$

Completeness

$\gamma_{p,\boldsymbol{v}}$: the geodesic with $\gamma_{p,\boldsymbol{v}}(0) = p$ and $\dot{\gamma}_{p,\boldsymbol{v}}(0) = \boldsymbol{v}$.

Definition valid for pseudo Riemannian case

- $\gamma_{p,\boldsymbol{v}}(t)$ is said to be <u>complete</u> if it is defined on $\mathbb{R}$.
- (M, g) is said to be <u>complete</u> if all geodesics are complete.

Example

The open submanifold $M := \mathbb{E}^n \setminus \{\mathbf{0}\}$ of the Euclidean space $\mathbb{E}^n$ is not complete. In fact, let $\boldsymbol{x} \in M$ and $\boldsymbol{v} := -\boldsymbol{x} \in \mathbb{E}^n = T_{\boldsymbol{x}}M$. Then the geodesic

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v} = (1-t)\boldsymbol{x}$$

is defined only on $(-\infty, 1)$.

Theorem (Hopf-Rinow's theorem)

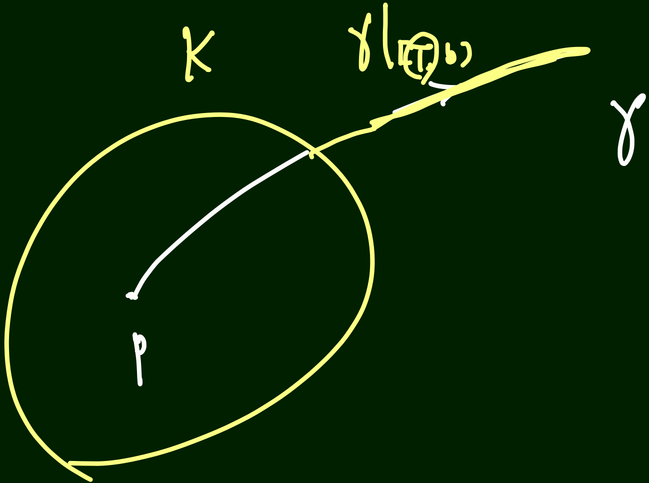
The following are equivalent:

1. (M, g) *is complete.* geodesic
2. $\exists p \in M$ *such that all geodesics emanating at* p *are complete.*
3. (M, d) *is a complete.* $\forall$ Cauchy sequences converge.
4. *Any bounded subset* D *of* M *is precompact.* $\bar{D}$: compact
5. ~~$\forall p, q \in M$, $\exists$ *the shortest geodesic joining* p *and* q.~~
6. *Any divergent path has infinite length.*

$$\gamma : [a, b) \to M : \text{divergent}$$

$$\Leftrightarrow \forall K : \text{compact subset of } M$$

$$\exists T \in \mathbb{R} \text{ s.t. } \gamma([T, b)) \subset M \setminus K$$


K
$\gamma|_{[T,b)}$
γ
p

Complete $\Rightarrow \exists$ shortest path

Exmaple: the Euclidean space

Example

The Euclidean space $\mathbb{E}^n$ is complete. In fact, $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v}$ is defined on $\mathbb{R}$.

the shortest path joining p & q :

the <u>line segment</u>.

Example

$$S^n := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$$

Then for each $\boldsymbol{x} \in S^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} S^n$,

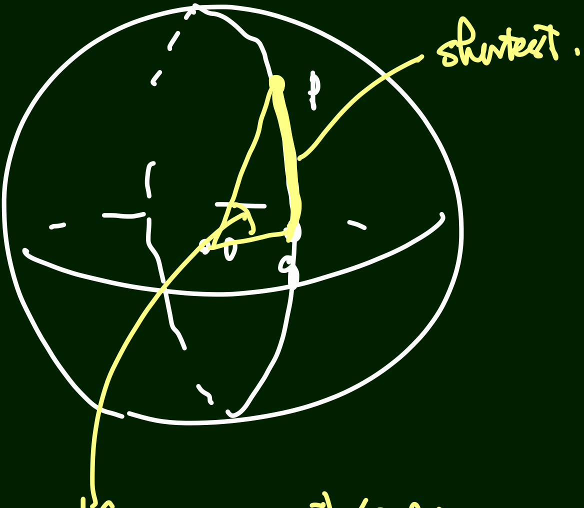
$$\underline{\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}'} \qquad \left(v = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{v} \right).$$

$t \in \mathbb{R}$

Rem S^n is complete because it is compact.

the shortest path joining p & q:

the shorter arc of the great circle passing through p & q.



$$d(p, q) = \cos^{-1}\langle p, q\rangle$$

Example: the hyperbolic space

Example

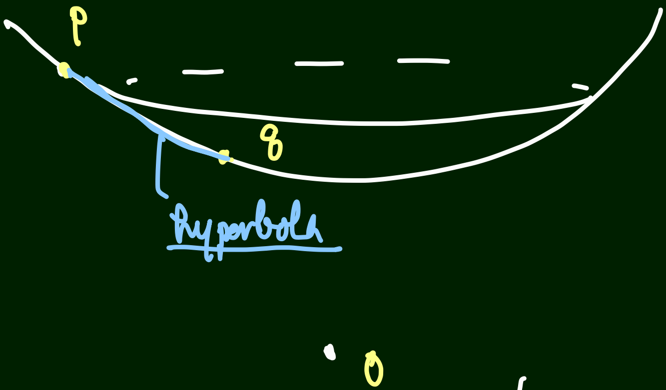
$$H^n := \left\{ \boldsymbol{x} = (x^0, \dots, x^{n+1}) \in \mathbb{E}_1^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0 \right\}.$$

Then for each $\boldsymbol{x} \in H^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} H^n$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' \qquad \left(v = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{v} \right).$$

p

complete.



Q $\quad d(p, q) = \cosh^{-1}\left(-\langle p, q\rangle\right)$

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Example

$$S^n_1 := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}.$$

Then for each $\boldsymbol{x} \in S^n_1$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} S^n_1$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle > 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle v, v \rangle = 0, \\ (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle < 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v} \rangle|^{1/2}$, $\boldsymbol{v}' := \dfrac{\boldsymbol{v}}{v}$.

(complete Lorentzian manifold)

$\mathbb{E}^n$ $\quad S^n(k)$ $\quad H^n(k)$ $\quad$ complete Riem. mfds

Goal

: complete Riem manifolds of constant sectional curvature

Classification

Ads.

Advanced Topics in Geometry F1 (MTH.B506) @M-143B (H119B)

June 13.	1.	Linear Ordinary Differential Equations
June 20.	2.	Integrability Condition
June 27.	3.	Differential Forms
July 04.	4.	Curvature
July 11.	5.	Sectional Curvature
July 18.	6.	Riemannian manifolds of constant sectional curvature
July 25.	7.	Fundamental Theorem for hypersurface theory