# Advanced Topics in Geometry E1 (MTH.B505)

Hopf-Rinow's theorem

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# Set up

$(M, g)$: a connected Riemannian manifold

# Length

## Definition

Let $\gamma\colon [a,b] \to M$ be a piecewize $C^1$-curve, where $[a,b]$ is a closed interval on $\mathbb{R}$. The integral

$$\mathcal{L}(\gamma) := \int_a^b |\dot{\gamma}(t)|\,dt$$

is called the length of $\gamma$.

# Distance

## Lemma

*For points* $p$, $q \in M$, *set*

$$d(p,q) := \inf\{\mathcal{L}(\gamma)\,;\,\gamma \in \mathcal{C}_{p,q}\}\colon M \times M \to \mathbb{R},$$
$$\left(\mathcal{C}_{p,q} := \left\{\gamma\colon [a,b] \to M\,;\, \begin{array}{l}\gamma \text{ is a piecewize } C^1\text{-curve} \\ \text{with } \gamma(a) = p \text{ and } \gamma(b) = q\end{array}\right\}\right)$$

*is a distance function on* $M$*, that is, it satisfies the axiom*

- $d(p,q) \geqq 0$ *for any* $p$, $q \in M$*. The equality holds iff* $p = q$.
- $d(p,q) = d(q,p)$.
- $d(p,q) + d(q,r) \geqq d(q,r)$

*of distance (compatible to the topology of* $M$*).*

$d$: the Riemannian distance with respect to $g$.

# The shortest geodesic

## Proposition

*For $p$, $q \in M$, a curve $\gamma \in \mathcal{C}_{p,q}$ satisfying $d(p, q) = \mathcal{L}(\gamma)$ is a pregeodesic.*

## Definition

The geodesic $\gamma \in \mathcal{C}_{p,q}$ satisfying $\mathcal{L}(\gamma) = d(p, q)$ is called the minimizing geodesic or the shortest geodesic joining $p$ and $q$.

# Completeness

$\gamma_{p,\boldsymbol{v}}$: the geodesic with $\gamma_{p,\boldsymbol{v}}(0) = p$ and $\dot{\gamma}_{p,\boldsymbol{v}}(0) = \boldsymbol{v}$.

## Definition

- $\gamma_{p,\boldsymbol{v}}(t)$ is said to be compolete if it is defined on $\mathbb{R}$.
- $(M, g)$ is said to be complete if all geodesics are complete.

## Example

The open submanifold $M := \mathbb{E}^n \setminus \{\boldsymbol{0}\}$ of the Euclidean space $\mathbb{E}^n$ is not complete. In fact, let $\boldsymbol{x} \in M$ and $\boldsymbol{v} := -\boldsymbol{x} \in \mathbb{E}^n = T_{\boldsymbol{x}}M$. Then the geodesic

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v} = (1 - t)\boldsymbol{x}$$

is defined only on $(-\infty, 1)$.

# Hopf-Rinow's Theorem

**Theorem (Hopf-Rinow's theorem)**

*The following are equivalent:*

1. $(M, g)$ *is complete.*
2. $\exists p \in M$ *such that all geodesics emanating at* $p$ *are complete.*
3. $(M, d)$ *is a complete.*
4. *Any bounded subset* $D$ *of* $M$ *is precompact.*
5. *Any divergent path has infinite length.*

# Exmaple: the Euclidean space

## Example

The Euclidean space $\mathbb{E}^n$ is complete. In fact, $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v}$ is defined on $\mathbb{R}$.

# Example: the Shpere

## Example

$$S^n := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$$

Then for each $\boldsymbol{x} \in S^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} S^n$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' \qquad \left( v = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{v} \right).$$

# Example: the hyperbolic space

## Example

$$H^n := \left\{ \boldsymbol{x} = (x^0, \dots, x^{n+1}) \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x^0 > 0 \right\}.$$

Then for each $\boldsymbol{x} \in H^n$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} H^n$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' \qquad \left( v = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{v} \right).$$

# Example: de Sitter space

## Example

$$S^n_1 := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}.$$

Then for each $\boldsymbol{x} \in S^n_1$ and $\boldsymbol{v} \in T_{\boldsymbol{x}} S^n_1$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle > 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle v, v \rangle = 0, \\ (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle < 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v} \rangle|^{1/2}$, $\boldsymbol{v}' := \frac{\boldsymbol{v}}{v}$.

# Ads.

Advanced Topics in Geometry F1 (MTH.B506) @M-143B (H119B)

| | |
|---|---|
| June 13. | 1. Linear Ordinary Differential Equations |
| June 20. | 2. Integrability Condition |
| June 27. | 3. Differential Forms |
| July 04. | 4. Curvature |
| July 11. | 5. Sectional Curvature |
| July 18. | 6. Riemannian manifolds of constant sectional curvature |
| July 25. | 7. Fundamental Theorem for hypersurface theory |