1 Inner products

Throughout this section, V denotes an n-dimensional vector space over $\mathbb{R}$ ($n < \infty$).

Bilinear forms

Definition 1.1. A *symmetric bilinear form* on the vector space V is a map $q\colon V \times V \to \mathbb{R}$ satisfying the following:

- For each fixed $\boldsymbol{x} \in V$, both $q(\boldsymbol{x}, \cdot)\colon V \ni \boldsymbol{y} \mapsto q(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}$ and $q(\cdot, \boldsymbol{x})\colon V \ni \boldsymbol{y} \mapsto q(\boldsymbol{y}, \boldsymbol{x}) \in \mathbb{R}$ are linear maps, and
- $q(\boldsymbol{x}, \boldsymbol{y}) = q(\boldsymbol{y}, \boldsymbol{x})$ holds for $\boldsymbol{x}$, $\boldsymbol{y} \in V$.

A symmetric bilinear form q is said to be *positive definite* if $q(\boldsymbol{x}, \boldsymbol{x}) > 0$ for any $\boldsymbol{x} \in V \setminus \{\boldsymbol{0}\}$.

Definition 1.2. An *inner product* on V is a map

$$\langle\ ,\ \rangle : V \times V \ni (\boldsymbol{x}, \boldsymbol{y}) \longmapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle \in \mathbb{R}$$

which is a positive definite symmetric bilinear form.

Example 1.3. We consider $\mathbb{R}^n$ the vector space consisting of n-dimensional *column* vectors. For an $n \times n$-symmetric matrix $A = (a_{ij})$ with real components,

$$q_A\colon \mathbb{R}^n \times \mathbb{R}^n \ni (\boldsymbol{x}, \boldsymbol{y}) \longmapsto \boldsymbol{x}^T A \boldsymbol{y} \in \mathbb{R}.$$

is a symmetric bilinear form, here ^{T} denotes the transposition.

Conversely, for each symmetric bilinear form q in $\mathbb{R}^n$, there exists a symmetric matrix A such that $q = q_A$. In fact, setting $a_{ij} := q(\boldsymbol{e}_i, \boldsymbol{e}_j)$, $A = (a_{ij})$ satisfies $q = q_A$, where $[\boldsymbol{e}_j]$ is the canonical basis of $\mathbb{R}^n$.

Definition 1.4. Let $(V, \langle\ ,\ \rangle)$ be an n-dimensional $\mathbb{R}$-vector space with inner product $\langle\ ,\ \rangle$. An *orthonormal basis* of $(V, \langle\ ,\ \rangle)$ is an n-tuple $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of elements of V satisfying

$$\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Proposition 1.5. (1) *An orthonormal basis of $(V, \langle\ ,\ \rangle)$ is a basis of V.*

(2) *For two orthonormal bases $[\boldsymbol{e}_j]$ and $[\boldsymbol{f}_j]$, there exists an orthogonal matrix P with*

$$[\boldsymbol{f}_1, \dots, \boldsymbol{f}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]P.$$

Proof. If $\boldsymbol{0} = x^1\boldsymbol{e}_1 + \dots + x^n\boldsymbol{e}_n$, $x_j = \langle 0, \boldsymbol{e}_j \rangle = 0$ for $j = 1, \dots, n$. Thus $[\boldsymbol{e}_j]$ is linearly independent. So noticing $\dim V = n$, we have (1).

Now we prove (2). If we set $p_{ij} := \langle \boldsymbol{f}_i, \boldsymbol{e}_j \rangle$ $(i, j = 1, \dots, n)$, we have $\boldsymbol{f}_i = p_{i1}\boldsymbol{e}_1 + \dots + p_{in}\boldsymbol{e}_n$, $(i = 1, \dots, n)$, in other words $[\boldsymbol{f}_1, \dots, \boldsymbol{f}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]P$ holds. Moreover, orthogonality of $[\boldsymbol{f}_i]$ and $[\boldsymbol{e}_j]$, it holds that

$$\delta_{ij} = \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle = \sum_{k=1}^{n} p_{ik}p_{jk} = ij \text{ component of } P^T P.$$

Hence P is an orthogonal matrix. □

Theorem 1.6. *[Existence of an orthonormal basis] For any n-dimensional $\mathbb{R}$-vector space $(V, \langle\ ,\ \rangle)$, an orthonormal basis of exists.*

Proof. Gram-Schmidt's orthogonalization. □

18. April, 2023.

Dual basis The vector space

$$V^* := \{\alpha\colon V \to \mathbb{R}\,;\, \textit{linear}\}$$

of linear maps from V to $\mathbb{R}$ is called the *dual space* of V.

Assume the inner product $\langle\ ,\ \rangle$ on V is given, and take an orthonormal basis $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ with respect to $\langle\ ,\ \rangle$. We set then

$$\omega^j\colon V \ni \boldsymbol{x} \longmapsto \omega^j(\boldsymbol{x}) := \langle \boldsymbol{e}_j, \boldsymbol{x}\rangle \in \mathbb{R}. \tag{1.1}$$

Proposition 1.7. *An n-tuple $[\omega^1,\dots,\omega^n]$ in (1.1) is a basis of V^*, called the* dual basis *of $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$.*

Proof. Assume $0 = a_1\omega^1 + \dots + a_n\omega_n$, where $0 \in V^*$ is the zero-map. Substituting $\boldsymbol{e}_j$ on the both side of it, we have $a_j = 0$. Hence $[\omega^j]$ is linearly-independent. On the other hand, for an arbitrary $\alpha \in V^*$, we set $a_j := \alpha(\boldsymbol{e}_j)$ $(j = 1,\dots,n)$. Then $a_1\omega^1 + \dots + a_n\omega^n = \alpha$, and hence V^* is spanned by $[\omega^j]$. □

Definition 1.8. For α, $\beta \in V^*$, a symmetric bilinear form

$$\alpha\beta\colon V \times V \ni (\boldsymbol{x},\boldsymbol{y}) \mapsto \frac{1}{2}(\alpha\boldsymbol{x})\beta(\boldsymbol{y}) + \alpha\boldsymbol{y})\beta(\boldsymbol{x})$$

is called the *symmetric product* of α and β. In particular, when $\beta = \alpha$, we denote $\alpha\alpha$ by α^2 for simplicity.

Proposition 1.9. *Let $(V, \langle\ ,\ \rangle)$ be an n-dimensional vector space V with inner product $\langle\ ,\ \rangle$. Take an orthonormal basis $[\boldsymbol{e}_j]$ and its dual basis $[\omega^j]$. Then*

$$\langle\ ,\ \rangle = (\omega^1)^2 + \dots + (\omega^n)^2.$$

Proof. Let $\boldsymbol{x}$, $\boldsymbol{y} \in V$. Then

$$\boldsymbol{x} = \sum_{i=1}^n \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle\, \boldsymbol{e}_i = \sum_{i=1}^n \omega^i(\boldsymbol{x})\boldsymbol{e}_i, \qquad \text{and} \qquad \boldsymbol{y} == \sum_{j=1}^n \omega^j(\boldsymbol{y})\boldsymbol{e}_j$$

holds. Thus,

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \sum_{i,j=1^n} \omega^i(\boldsymbol{x})\omega^j(\boldsymbol{y})\,\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \sum_{i=1}^n \omega^i(\boldsymbol{x})\omega^i(\boldsymbol{y}) = \sum_{i=1}^n (\omega^i)^2(\boldsymbol{x},\boldsymbol{y}). \qquad \square$$

The Euclidean vector space Throughout this lecture, we consider $\mathbb{R}^n$ as a set of n-dimensional *column* vector. We set

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle := \boldsymbol{x}^T\boldsymbol{y} = \sum_{j=1}^n x^j y^j, \qquad (\boldsymbol{x} = (x^1,\dots,x^n)^T, \boldsymbol{y} = (y^1,\dots,y^n)^T).$$

Then $\langle\ ,\ \rangle$ is an inner product, which is called the *canonical inner product.*

Definition 1.10. A pair $\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$ is called the *Euclidean vector space.*

Similarly, we consider $\mathbb{R}^{n+1}$, and set

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle_L := -x^0y^0 + \sum_{j=1}^n x^j y^j (\boldsymbol{x} = (x^0, x^1,\dots,x^n)^T, \boldsymbol{y} = (y^0, y^1,\dots,y^n)^T),$$

and call it the *canonical Lorentz-Minkowski inner product.*

Definition 1.11. A pair $\mathbb{L}^{n+1} := (\mathbb{R}^{n+1}, \langle\ ,\ \rangle_L)$ is called the *Lorentz-Minkowski vector space.*

Appendix: A Review of Undergraduate Linear Algebra.

Definition 1.12.

- A square matrix P of real components is said to be an *orthogonal matrix* if $P^T P = PP^T = I$ holds, where P^T denotes the transposition of P and I is the identity matrix.
- A square matrix $A = (a_{ij})$ is said to be (real) *symmetric matrix* if $A^T = A$, which is equivalent to that $a_{ij} = a_{ji}$, holds.

Fact 1.13.

- *The eigenvalues of a real symmetric matrix are real numbers, and the dimension of the corresponding eigenspace coincides with the multiplicity of the eigenvalue.*
- *Real symmetric matrices can be diagonalized by orthogonal matrices. In other words, for each real symmetric matrix A, there exists an orthogonal matrix P satisfying*

$$P^{-1}AP = P^T AP = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

where $\operatorname{diag}(\dots)$ *denotes the diagonal matrix with diagonal components "$\dots$". In particular,* $\{\lambda_1, \dots, \lambda_n\}$ *are the eigenvalues of A counted with their multiplicity.*

Exercises

1-1 Let $\langle\ ,\ \rangle$ be an inner product of $\mathbb{R}^2$ defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^T A \boldsymbol{y} \qquad A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix},$$

where a is a real number with $|a| < 1$.

- Find an orthonormal basis $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ with respect to $\langle\ ,\ \rangle$.
- Find row vectors $\hat{\omega}^j$ $(j = 1, 2)$ such that the dual basis $[\omega^j]$ of $[\boldsymbol{e}_j]$ is expressed as

$$\omega^j(\boldsymbol{x}) = \hat{\omega}^j \boldsymbol{x} \qquad (j = 1, 2).$$

1-2 Let $\mathbb{L}^3$ be the 3-dimensional Lorentz-Minkowski vector space, and fix $\boldsymbol{x} \in \mathbb{L}^3$ with $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -1$. Take the "orthogonal complement"

$$W := \boldsymbol{x}^\perp = \{\boldsymbol{y} \in \mathbb{L}^3\,;\, \langle \boldsymbol{x}, \boldsymbol{y} \rangle\}.$$

- Show that W is an 2-dimensional linear subspace of $\mathbb{L}^3$.
- Show that the restriction of $\langle\ ,\ \rangle_L$ to $W \times W$ is a (positive definite) inner product of W.