2 Riemannian manifolds

Manifolds

Definition 2.1. Let M be a topological space, and fix a positive integer n. A pair (U,φ) of an open set $U \subset M$ and $\varphi\colon U \to \mathbb{R}^n$ is said to be an (n-dimensional) *chart* or a *local coordinate system*, if φ is homeomorphism of U to $\varphi(U)$. Two charts (U,φ) and (V,ψ) are said to be *adapted* if $U\cap V=\emptyset$ or $\varphi\circ\psi^{-1}\colon \mathbb{R}^n\supset\psi(U\cap V)\to\varphi(U\cap V)\subset\mathbb{R}^n$ is a diffeomorphism.

Example 2.2. A pair $(\mathbb{R}^n,\mathrm{id})$ is a chart of $\mathbb{R}^n$, where id is the identity map.

The polar coordinate system (U,φ) as in Example 2.29 is an adapted chart with $(\mathbb{R}^2,\mathrm{id})$.

Definition 2.3. An n-dimensional *smooth manifold* [1], or simply *manifold* is a Housdorff topological space M with second axiom of countability, endowed with a family $\mathcal{A} := \{(U_\lambda,\varphi_\lambda)\,;\,\lambda\in\Lambda\}$ of n-dimensional charts which are mutually adapted, called the *atlas*, [2]satisfying $\cup_{\lambda\in\Lambda}U_\lambda = M$.

Example 2.4. For each positive integer n, $\mathbb{R}^n$ is an n-manifold with atlas $\mathcal{A} := \{(\mathbb{R}^n,\mathrm{id})\}$, which is called the n-dimensional *affine space*.

Tangent space Let M be an n-dimensional manifold with atlas $\mathcal{A}$. A function $f\colon M\to\mathbb{R}$ is said to be *smooth* if $f\circ\varphi^{-1}\colon\mathbb{R}^n\supset\varphi(U)\to\mathbb{R}$ is smooth for any chart in $\mathcal{A}$. A map $\boldsymbol{f}\colon M\to\mathbb{R}^k$ is said to be smooth if all components of $\boldsymbol{f}$ are all smooth function on M.

Fact 2.5. *A function $f\colon M\to\mathbb{R}$ is smooth if for each point $p\in M$, <u>there exists a chart</u> $(U,\varphi)\in\mathcal{A}$ with $p\in U$ and $f\circ\varphi$ is smooth.*

Fact 2.6. *The set $\mathcal{F}(M)$ of smooth functions on M can be considered as an algebra over $\mathbb{R}$ by natural addition and multiplication.*

For a chart $(U,\varphi)\in\mathcal{A}$, we write $\varphi: U\ni q\mapsto\varphi(q) = \bigl(x^1(q),\dots,x^n(q)\bigr)\in\mathbb{R}^n$. Then $x^j\colon U\to\mathbb{R}$ is a smooth function for each $j=1,\dots,n$. Such x^j's are called the *coordinate function* with respect to the chart. . If we fix $\varphi=(x^1,\dots,x^n)$, we write $f\circ\varphi^{-1}(x^1,\dots,x^n)$ by $f(x^1,\dots,x^n)$, for a sake of simplicity.

Definition 2.7. Fix a point $p\in M$. A *tangent vector* of M at p is an $\mathbb{R}$-linear map $X_p\colon\mathcal{F}(M)\to\mathbb{R}$ satisfying the "Leibniz rule": $(X_p)(fg) = f(p)(X_p)(g)+g(p)X_p(f)$. We denote by T_pM the vector space consisting of the tangent vectors at p, and call the *tangent space* of M at p.

Fact 2.8. *Fix a chart $\bigl(U,\varphi=(x^1,\dots,x^n)\bigr)$ containing p. Then $\bigl(\frac{\partial}{\partial x^j}\bigr)_p : \mathcal{F}(M)\ni f\mapsto\frac{\partial f}{\partial x^j}(p)\in\mathbb{R}$ is an element of T_pM. Moreover, $\Bigl[\bigl(\frac{\partial}{\partial x^1}\bigr)_p,\dots,\bigl(\frac{\partial}{\partial x^n}\bigr)_p\Bigr]$ is a basis of T_pM. In particular T_pM is an n-dimensional vector space.*

Let $\widetilde{M}$ be another manifold of dimension m. A map $f\colon M\to\widetilde{M}$ is said to be *smooth* if $\psi\circ f$ is smooth for an arbitrary chart (V,ψ) of $\widetilde{M}$. In particular, a map $\gamma\colon I\to M$ defined on an open interval $I\subset\mathbb{R}$ is a *smooth curve* if for each $t_0\in I$, $\varphi\circ\gamma$ is a smooth map on a neighborhood of t_0 into $\mathbb{R}^n$, where (U,φ) is a chart containing $\gamma(t_0)$.

Fact 2.9. *For a smooth curve $\gamma(t)$ on M with $\gamma(0)$, $\mathcal{F}(M)\ni f\mapsto\frac{d}{dt}f\circ\gamma(0)\in\mathbb{R}$ is an element of T_pM, denoted by $\dot\gamma(0)$. Conversely, any $X_p\in T_pM$, there exists a smooth curve $\gamma(t)$ with $\gamma(0)=p$, such that $\dot\gamma(0)=X_p$. In this sense, X_p can be interpreted as a* directional derivative.

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[1]The word "smooth" is used as a synonym of "of class C^∞" in this lecture.

[2]Usually the atlas $\mathcal{A}$ of a given manifold M is assumed to be *maximal*, that is, any chart (U,φ) adapted with arbitrary chart in $\mathcal{A}$ is an element of $\mathcal{A}$.

Example 2.10. Let $\mathbb{R}^n$ be an n-dimensional affine space as in Example 2.4. Then $\mathcal{F}(\mathbb{R}^n)$ is the set of C^∞-functions of n-variables. A *directional derivative* of f at p in the direction $X \in \mathbb{R}^n$ as $\frac{d}{dt}\big|_{t=0} f(p+tX)$ is identified with an element of $T_p\mathbb{R}^n$. Thus, $T_p\mathbb{R}^n$ is identified $\mathbb{R}^n$ itself (considered as a vector space).

Definition 2.11. The disjoint sum $TM := \cup_{p\in M} T_pM$ is called the *tangent bundle* of M. Define the *projection* $\pi\colon TM \ni X \to \pi(X) \in M$, where $p = \pi(X)$ is the unique point on M such that $X \in T_pM$. For each chart $(U, \varphi = (x^1, \dots, x^n))$ of M, set $\Phi\colon \pi^{-1}(U) \ni X \mapsto (\varphi(\pi(X)), X^1, \dots, X^n) \in \mathbb{R}^{2n}$, where $X = \sum X^j (\partial/\partial x^j)_{\pi(X)}$. Then a structure of $2n$-manifold on TM such that (1) π is a smooth map (U, Φ) is an adapted chart for each chart (U, φ) on M.

Example 2.12. Since $T_p\mathbb{R}^n$ is identified with $\mathbb{R}^n$ (Example 2.10), the tangent bundle of the affine space $\mathbb{R}^n$ is the product $\mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.13. A (smooth) *vector field* on M is a smooth map $X\colon M \to TM$ satisfying $\pi \circ X = \mathrm{id}_M$, where id_M is the identity map of M.

Example 2.14. Identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$, a vector field of an affine space $\mathbb{R}^n$ is regarded as a smooth map $X\colon \mathbb{R}^n \to \mathbb{R}^n$.

We denote by $\mathfrak{X}(M)$ the set of vector fields on M. For a function $f \in \mathcal{F}(M)$ and a vector field $X \in \mathfrak{X}(M)$, the (pointwise) scalar multiplication fX is also a vector field on M. Thus, $\mathfrak{X}(M)$ has a structure of $\mathcal{F}(M)$-module.

Submanifolds Let M be an n-manifold. A smooth map $\boldsymbol{f}\colon M \to \mathbb{R}^k$ is of *rank* r at $p \in M$ if there exists a chart (U, φ) on M containing p such that the Jacobian matrix of $\boldsymbol{f} \circ \varphi^{-1}$ at $\varphi(p)$ is of rank r. In particular, a map $\boldsymbol{f}$ is said to be an *immersion* if it is of rank $n = \dim M$ for all $p \in M$.

Example 2.15. Let $M = (-\pi, \pi) \times \mathbb{R}$ and set

$$\boldsymbol{f}\colon M \ni (u, v) \longmapsto (\operatorname{sech} v \cos u, \operatorname{sech} v \sin u, v - \tanh v)^T \in \mathbb{R}^3.$$

Then $\boldsymbol{f}$ is of rank 2 if $v \neq 0$, and of rank 1 where $v = 0$.

Definition 2.16. A subset M of $\mathbb{R}^k$ (endowed with the topology induced to the canonical topology of $\mathbb{R}^k$) is called a *submanifold* of $\mathbb{R}^k$ if there exists a structure of manifold on M such that the inclusion map M is an immersion. [3].

Example 2.17. A open subset $U \subset \mathbb{R}^k$ is a k-dimensional submanifold of $\mathbb{R}^k$.

Fact 2.18 (Implicit Function Theorem)**.** *Let $\boldsymbol{F}\colon \mathbb{R}^{n+r} \to \mathbb{R}^r$ be a smooth map, where n and r are positive numbers, and assume $M := \boldsymbol{F}^{-1}(\boldsymbol{0}) = \{p \in \mathbb{R}^{n+r}\,;\, \boldsymbol{F}(p) = \boldsymbol{0}\}$ is not empty. Then M is an n-dimensional submanifold of $\mathbb{R}^{n+r}$ provided $\boldsymbol{F}$ is of rank r on M. In this case, the tangent space T_pM $(\subset \mathbb{R}^{n+r} = T_p\mathbb{R}^{n+r})$ can be identified as the kernel of the Jacobian matrix $d\boldsymbol{F}(p)$ of $\boldsymbol{F}$ at p.*

Example 2.19 (Spheres)**.** Let k be a positive number and set $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$F(\boldsymbol{x}) := F(x^0, \dots, x^n) = \left(\sum j = 0^n (x^j)^2\right) - \frac{1}{k} = \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \frac{1}{k},$$

where $\langle\ ,\ \rangle$ is the canonical inner product of $\mathbb{R}^{n+1} = \mathbb{E}^{n+1}$. Then $dF = 2(x^0, \dots, x^n)$ vanishes if and only if $\boldsymbol{x} = \boldsymbol{0}$ where $F(\boldsymbol{0}) = -1/k \neq 0$. Hence $S^n(k) := F^{-1}(0)$ is n-dimensional submanifold of $\mathbb{R}^{n+1}$, which is the n-dimensional *sphere*.The tangent space of $S^n(k)$ at $\boldsymbol{x}$ is

$$T_{\boldsymbol{x}} S^n(k) = \boldsymbol{x}^\perp = \{\boldsymbol{v} \in \mathbb{R}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\}.$$

[3]More generally, a notion of sabmanifolds in a manifold $\widetilde{M}$, because the rank of $\boldsymbol{f}\colon M \to \widetilde{M}$ can be defined by using coordinate function on $\widetilde{M}$.

Thus, the tangent bundle is expressed as a submanifold of $\mathbb{R}^{2(n+1)}$ as

$$TS^n(k) = \{(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1/k, \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{2(n+1)}.$$

Example 2.20. Consider the Lorentz-Minkowski inner product $\langle\ ,\ \rangle_L$ on $\mathbb{R}^{n+1} = \mathbb{L}^{n+1}$, and set

$$F_r(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L - q,$$

where r is a real constant. Then $M_q := F_q^{-1}(0)$ is a submanifold of $\mathbb{R}^{n+1}$ if $q \neq 0$, and the tangent space of M_q at $\boldsymbol{x}$ is

$$T_{\boldsymbol{x}} M_q = \boldsymbol{x}^{\perp} = \{\boldsymbol{v} \in \mathbb{R}^{n+1} \,;\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle_L = 0\}.$$

When $q < 0$, $\boldsymbol{x} = (x^0, \dots, x^n)^T \in M_q$ satisfies $|x^0| \geqq 1/\sqrt{q}$. Thus M_q is not connected. We denote the connected component as

$$H^n(k) := \{(x^0, \dots, {}^{n_1})^T \in M_q \,;\, x^0 > 0\} \qquad (k = 1/q).$$

When $q > 0$ $M_q := S_1^{n-1}(k))$ $(k = 1/q)$is connected submanifold.

When $q = 0$, M_0 is called the *cone* or *light cone* which has a *singularity* at $\mathbf{0}$.

Riemannian manifolds

Definition 2.21. A *Riemannian metric* g on an n-manifold M is a correspondence $p \mapsto g_p$ of p to an inner product g_p of T_pM, which satisfies the smoothness condition, that is,

$$g(X, Y) : M \ni p \mapsto g_p(X_p, y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields (X, Y).

Example 2.22. [the Euclidean space] Identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$, the Euclidean inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ induces a Riemannian metric of $\mathbb{R}^n$. $\mathbb{E}^n := (\mathbb{R}^n, \langle\ ,\ \rangle)$ is called the *Euclidean space*.

Example 2.23. Let M be an n-dimensional submanifold of $\mathbb{E}^{n+r}$. Since the restriction of the inner product $\langle\ ,\ \rangle$ of $T_{\boldsymbol{x}}\mathbb{E}^{n+1} = \mathbb{E}^{n+1}$ to the tangent space $T_{\boldsymbol{x}}M \subset \mathbb{E}^{n+1}$ is positive definite, it defines a Riemannian metric of M. Such a metric is called the *induced metric* from $\mathbb{E}^{n+r}$.

Example 2.24. [the sphere] The sphere $S^n(k) \subset \mathbb{R}^{n+1}$ of *curvature* k is a submanifold of the Euclidean space with induced metric from $\mathbb{E}^{n+1}$.

The hyperbolic space Let $H^n(k)$ $(k < 0)$ be as in Example 2.20. For each position vector $\boldsymbol{x} \in H^n(k) \in \mathbb{R}^{n+1}$, $\boldsymbol{e} := \boldsymbol{x}/\sqrt{k}$ satisfies $\langle \boldsymbol{e}, \boldsymbol{e} \rangle = -1$. Then the restriction of the Lorentz-Minkowski inner product $\langle\ ,\ \rangle_L$ to the tangent space $T_{\boldsymbol{x}}H^n(k) = \boldsymbol{x}^{\perp} = \boldsymbol{e}^{\perp}$ is positive definite, as seen in Exercise 1-2. Thus, $\langle\ ,\ \rangle_L$ induces a Riemannian metric on $H^n(k)$. The Riemannian manifold obtained in this way is called the *hyperbolic space* of curvature k.

Appendix: Diffeomorphisms

Definition 2.25. Let U and V be open subsets of $\mathbb{R}^n$. A *diffeomorphism* from U to V is a bijection $\varphi\colon U \to V$ of class C^∞ whose inverse $\varphi^{-1}\colon V \to U$ is also of class C^∞.

Example 2.26.

- A map $\varphi\colon \mathbb{R} \supset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ni x \mapsto \tan x \in \mathbb{R}$ is a diffeomorphism.
- A bijection $\psi\colon \mathbb{R} \ni x \mapsto x^3 \in \mathbb{R}$ is not a diffeomorphism, because $\psi^{-1}(y) = \sqrt[3]{y}$ is not differentiable at $y = 0$.

Theorem 2.27. *Let $\varphi\colon U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset U of $\mathbb{R}^n$, and write $\varphi(x^1,\dots,x^n) = \bigl(y^1(x^1,\dots,x^n),\dots,y^n(x^1,\dots,x^n)\bigr)$. If the Jacobian $J_\varphi := \det(\frac{\partial y^j}{\partial x^k})_{j,k=1,\dots,n}$ does not vanish at $p \in U$, there exists a neighborhood V $(\subset U)$ of p such that $\varphi|_v\colon V \to \varphi(V)$ is a diffeomorphism.*

Corollary 2.28. *Let U and V be open sets of $\mathbb{R}^n$. A C^∞-bijection $\varphi\colon U \to V$ is a diffeomorphism if its Jacobian J_φ does not vanish on U.*

Example 2.29. Let $V := (0,\infty)\times(-\pi,\pi) \subset \mathbb{R}^2$ and define $\psi\colon V \to \mathbb{R}^2$ by $\psi(r,\theta) = (r\cos\theta, r\sin\theta)$. Then ψ is a bijection from V to $U := \mathbb{R}^2\setminus\{(x,0)\,;\, x \leqq 0\}$. Since the Jacobian $J_\psi = r \neq 0$, Corollary 2.28 implies that the map ψ is diffeomorphism. Hence there exists the inverse $\varphi := \psi\colon U \to V \subset \mathbb{R}^2$, which is called the *polar coordinate system* of the plane.

Exercises

2-1 Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2 < 1\}$, and set

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3.$$

- For each $(u,v) \in D$,
- Show that $\boldsymbol{f}$ is a bijection from D to $H^3(-1)$.
- Compute $\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle$, $\langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle$ and $\langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle$.
- For each $(u,v) \in D$, find an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ of $T_{\boldsymbol{x}}H^3(-1)$, where $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

2-2 Fix an $(n+1)\times(n+1)$-orthogonal matrix A and set

$$\boldsymbol{\varphi}\colon S^n(k) \ni \boldsymbol{x} \mapsto A\boldsymbol{x} \in \mathbb{R}^{n+1},$$

where k is a positive number. Fix $\boldsymbol{x} \in S^n(k)$ and take a smooth curve $\gamma(t)$ on $S^n(k)$ such that $\gamma(0) = \boldsymbol{x}$ and set $\boldsymbol{v} := \dot\gamma(0) \in T_{\boldsymbol{x}}S^n(k)$.

- Show that $\boldsymbol{\varphi}$ induces a bijection from $S^n(k)$ into $S^n(k)$.
- Show that $\boldsymbol{\varphi}_*\boldsymbol{v} := \frac{d}{dt}\Big|_{t=0}\boldsymbol{\varphi}\circ\gamma = A\boldsymbol{v}$.
- Verify that $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \boldsymbol{\varphi}_*\boldsymbol{v}, \boldsymbol{\varphi}_*\boldsymbol{v}\rangle$.