# 3 Pseudo Riemannian manifolds

**Indefinite inner product** Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$. A symmetric bilinear form $\langle\ ,\ \rangle$ on $V$ is said to be *non-degenerate* if

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0 \qquad \text{for all } \boldsymbol{y} \in V \qquad \Leftrightarrow \qquad \boldsymbol{x} = \boldsymbol{0}.$$

Recall that $\langle\ ,\ \rangle$ is said to be *positive definite* (resp. *negative definite*) if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$ (resp. $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$) for all $\boldsymbol{x} \in V \setminus \{\boldsymbol{0}\}$. We often call a non-degenerate symmetric bilinear form an *inner product*. In such a context, an inner product in the sense of Definition 1.2 is called a *positive definite inner product*. An inner product neither positive nor negative definite is said to be *indefinite*.

**Example 3.1.** Let $n \geqq 1$ and $r \geqq 0$ be integers, and set

$$\mathbb{E}^{m+r}_r := (\mathbb{R}^{m+r}, \langle\ ,\ \rangle), \qquad \text{where} \qquad \langle \boldsymbol{x}, \boldsymbol{y}\rangle = -\left(\sum_{j=1}^{r} x^j y^j\right) + \left(\sum_{l=r+1}^{m+r} x^l y^l\right).$$

Then $\langle\ ,\ \rangle$ is a non-degenerate symmetric bilinear form on the vector space $\mathbb{R}^{m+r} = \mathbb{E}^{m+r}_r$.

When $r = 0$ (resp. $m = 0$), $\langle\ ,\ \rangle$ is positive (resp. negative) definite, otherwise it is indefinite.

It is obvious that

**Lemma 3.2.** *A positive (negative) definite symmetric bilinear form is non-degenerate.*

**Example 3.3.** Let $W \subset \mathbb{R}^3 = \mathbb{E}^3_1$ be a linear subspace given by

$$W = \boldsymbol{x}^\perp = \{\boldsymbol{v}\,;\, \langle \boldsymbol{x}, \boldsymbol{v}\rangle = 0\} \qquad (\boldsymbol{x} \neq \boldsymbol{0}).$$

Since the map $\boldsymbol{v} \mapsto \langle \boldsymbol{x}, \boldsymbol{v}\rangle$ is a liner map of rank 1, its kernel $W$ is a two dimensional subspace, called the *orthogonal complement* of $\boldsymbol{x}$.

Set $\boldsymbol{x} := (1,1,0)^T$. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$, $\boldsymbol{x} \in W = \boldsymbol{x}^\perp$ and the restriction $\langle\ ,\ \rangle|_{W\times W}$ is degenerate.

Take a linear subspace $W \subset V$. It is obvious that the restriction $\langle\ ,\ \rangle|_{W\times W}$ of a symmetric bilinear form $\langle\ ,\ \rangle$ on $V$ is a symmetric bilinear form on $W$.

**Lemma 3.4.** *If $\langle\ ,\ \rangle$ is positive (negative) definite, so is its restriction $\langle\ ,\ \rangle|_{W\times W}$.*

Let $\langle\ ,\ \rangle$ be a (not necessarily definite) inner product on an $n$-dimensional vector space $V$. Set

$$\begin{aligned} m &:= \max\{\dim W\,;\, W \subset V: \text{ a linear subspace}, \langle\ ,\ \rangle|_{W\times W} \text{ is positive definite}\}, \\ r &:= \max\{\dim W\,;\, W \subset V: \text{ a linear subspace}, \langle\ ,\ \rangle|_{W\times W} \text{ is negative definite}\}. \end{aligned} \tag{3.1}$$

**Lemma 3.5.** *The subspace $W_+$ (resp. $W_-$) of $V$ of dimension $m$ (resp. ) spans $V$, that is, $V = W_+ \oplus W_-$. In particular $m + r = n$.*

*Proof.* Since $W_- \cap W_+ = \{\boldsymbol{0}\}$, $m + r = \dim W_+ + \dim W_- \leqq n = \dim V$.

Take a basis $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ on $V$ and set $A := (\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle)_{i,j=1,\dots,n}$. Since $A$ is symmetric, its eigenvalues $(\lambda_1, \dots, \lambda_n)$ are all real. Moreover, one can take another basis $[\boldsymbol{w}_j]$ of $V$ such that $A\boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j$ for $j = 1, \dots, n$. Since $\lambda_j = 0$ implies $\langle \boldsymbol{w}_j, \boldsymbol{w}_j\rangle = 0$, all eigenvalues are non-zero. So we may assume $\lambda_1, \dots, \lambda_{r'}$ are negative, and $\lambda_{r'+1}, \dots, \lambda_n$ are positive, without loss of generality. Since the restriction of $\langle\ ,\ \rangle$ to the subspace $W'_-$ spanned by $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}'_r\}$ is negative definite, it holds that $r' \leqq r$. On the other hand, Since the restriction of $\langle\ ,\ \rangle$ to the subspace $W'_+$ spanned by $\{\boldsymbol{w}_{r'+1}, \dots, \boldsymbol{w}_n\}$ is positive definite, it holds that $n - r' \leqq m$.

Hence $r' \leqq r$, $n - r' \leqq m$, $m + r \leqq n$, then we have $r = r'$ and $m = n - r'$. □

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**Definition 3.6.** The pair $(m, r)$ defined in (3.1) is called the *signature* of $\langle\ ,\ \rangle$.

In particular, the inner product of signature $(n, 0)$ (resp. $(0, n)$) is positive (resp. negative) definite.

**Example 3.7.** The inner product of $\mathbb{E}^{m+r}_r$ has the signature $(m, r)$.

**Proposition 3.8.** *Let $\langle\ ,\ \rangle$ be an inner product of signature $(m, r)$ on an $n$-dimensional vector space $V$. For $\boldsymbol{x} \in V$ with $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$ (resp. $< 0$), the restriction of $\langle\ ,\ \rangle$ to $W := \boldsymbol{x}^\perp$ has signature $(m-1, r)$ (resp. $(m, r-1)$).*

*Proof.* Assume $\langle \boldsymbol{x}, \boldsymbol{x}\rangle < 0$ and let $(m', r')$ be the signature of $W = \boldsymbol{x}^\perp$. Since $W$ is of dimension $n-1$, $m' + r' = n - 1$ holds.

Take a subspace $W'_- \subset W$ on which $\langle\ ,\ \rangle$ is negative definite and $\dim W'_- = r'$, and a subspace $W'_+ \subset W$ on which $\langle\ ,\ \rangle$ is positive and $\dim W'_+ = m' = n - 1 - r'$.

Since $\langle\ ,\ \rangle$ is negative definite on $W'_- \oplus \mathbb{R}\boldsymbol{x}$, $r' + 1 \leqq r$ holds. On the other hand, since $\langle\ ,\ \rangle$ is positive on $W'_+ \subset V$, we have $m' = n - 1 - r' \leqq n - r$. Hence $r = r' + 1$ and the conclusion follows. □

### Pseudo Riemannian manifolds

**Definition 3.9.** A *pseudo Riemannian metric* $g$ of signature $(m, r)$ on a connected $n$ $(= m + r)$-manifold $M$ is a correspondence $p \mapsto g_p$ of $p$ to an inner product $g_p$ of signature $(m, r)$ on $T_pM$, which satisfies the smoothness condition, that is,

$$g(X, Y) : M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$$

is a smooth function for each pair of sooth vector fields $(X, Y)$.

A connected $n$-manifold $M$ endowed with a pseudo Riemannian metric $g$ is called a *pseudo Riemannian manifold*

A pseudo Riemannian manifold of signature $(n, 0)$ is nothing but a Riemannian manifold. A pseudo Riemannian manifold of signature $(n-1, 1)$ is called a *Lorentzian manifold*.

**Example 3.10.** Similar to the case of Euclidean space, The pseudo Euclidean vector space $\mathbb{E}^{m+r}_r$ induces a pseudo Riemannian metric on $\mathbb{R}^{m+r}$ of signature $(m, r)$. As a result, $\mathbb{E}^{m+r}_r$ is considered as a pseudo Riemannian manifold of signature $(m, r)$, called the *pseudo Euclidean space*. In particular $\mathbb{E}^{n+1}_1$ is called the *Lorentz Minkowski* $(n + 1)$*-space*.

**Example 3.11.** Let $a$ be a real number, and set

$$M^n(a) := \{\boldsymbol{x} \in \mathbb{E}^{n+1}_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = a\}.$$

When $a = 1/k > 0$, the connected component of $M^n(a)$ is the hyperbolic space of curvature $k$, as defined in Section 2.

When $a = -1/k < 0$, $M^n(a)$ is a connected submanifold of $\mathbb{E}^{n+1}_1$. Similar to the hyperbolic space, the tangent space $T_{\boldsymbol{x}}M^n(a)$ is the orthogonal complement $\boldsymbol{x}^\perp$ of the position vector $\boldsymbol{x}$. So, by Proposition 3.8, the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{E}^{n+1}_1$ to $T_{\boldsymbol{x}}M^n(r)$ has signature $(n-1, 1)$. The Lorentzian manifold obtained in this way is called the $n$-dimensional *de Sitter space* of curvature $k$ $(> 0)$.

Setting $r = 0$, $M^n(0)$ (called the *lightcone*) has the singularity at the origin. On the submanifold $M^n(0)\setminus\{0\}$, the tangent space at $\boldsymbol{x}$ is the orthogonal complement. Since $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0$, $\boldsymbol{x} \in T_{\boldsymbol{x}}M^n(0)$, and induced inner product on $T_{\boldsymbol{x}}M^n(0)$ degenerates.

**Example 3.12.** Let $a = 1/k$ be a negative real number, and set

$$M^n(a) := \{\boldsymbol{x} \in \mathbb{E}^{n+1}_2\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = a\}.$$

Then $M^n(a)$ is a submanifold of $\mathbb{E}^{n+1}_2$, and the induced metric has signature $(n-1, 1)$, that is, $M^n(a)$ is a Lorentzian manifold, called the *anti de Sitter space*.

***Exercises***

**3-1** Let $\mathrm{O}(2,1)$ be the set of $3\times 3$-matrices satisfying

$$\mathrm{O}(2,1) := \left\{ A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \in \mathrm{M}_3(\mathbb{R})\,;\, A^T Y A = Y \right\} \qquad \left( Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

- Show that $|\det A| = 1$ for $A \in \mathrm{O}(2,1)$.
- Show that $|a_{00}| \geqq 1$ for $A = (a_{ij})$.
- Show that the liner transformation induced by $A \in \mathrm{O}(2,1)$ preserves the inner product $\langle\ ,\ \rangle$ of $\mathbb{E}^3_1$.
- $\mathrm{SO}_+(2,1) := \{A = (a_{ij}) \in \mathrm{O}(2,1)\,;\, \det A = 1, a_{00} \geqq 1\}$ induces a bijection from the hyperbolic space $H^2(k) \subset \mathbb{E}^3_1$ to itself, where $k < 0$.

**3-2** Let $D := \{(u,v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\}$, and set

$$\boldsymbol{f} : D \ni (u,v) \mapsto \frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2, 2u, 2v\bigr) \in \mathbb{L}^3 = \mathbb{E}^3_1$$

as in Problem 2-1, and take an orthonormal basis $[\boldsymbol{e}_1(u,v), \boldsymbol{e}_2(u,v)]$ of $T_{\boldsymbol{x}}H^3(-1)$, where $\boldsymbol{x} = \boldsymbol{f}(u,v)$.

- Verify that, for each $(u,v) \in D$, $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$ is a basis of $\mathbb{R}^3$, where $\boldsymbol{e}_0 = \boldsymbol{f}$.
- Express the derivatives $(\boldsymbol{e}_j)_u$ and $(\boldsymbol{e}_j)_v$ $(j = 0,1,2)$ as linear combinations of $[\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2]$.