4 Riemannian connection for submanifolds in (pseudo) Euclidean spaces

Lie bracket Let M be an n-dimensional manifold, and denote by $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ the set of smooth functions and the set of smooth vector fields on M.

Take a vector field $X \in \mathfrak{X}(M)$ and fix a point $p \in M$. Then $X_p \in T_pM$ is a tangent vector in the sense of Definition 2.7, and hence for each $f \in \mathcal{F}(M)$, $Xf \colon M \ni p \mapsto X_pf \in \mathbb{R}$ is a smooth function. Take another vector field Y, then we obtain a function $Y(Xf)$ on M.

Express X and Y on a local chart $(U; x^1, \dots, x^n)$ as

$$X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}. \tag{4.1}$$

Then by Fact 2.8, we have the local expression of $Y(Xf)$ as

$$\begin{aligned} Xf &= \sum_{l=1}^n X^l \frac{\partial f}{\partial x^l}, \\ Y(Xf) &= \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}\left(\sum_{l=1}^n X^l \frac{\partial f}{\partial x^l}\right) = \sum_{j,l=1}^n Y^j \left(X^l \frac{\partial^2 f}{\partial x^j \partial x^l} + \frac{\partial X^l}{\partial x^j}\frac{\partial f}{\partial x^l}\right) \end{aligned} \tag{4.2}$$

which includes the second derivative of f. Thus, $f \mapsto Y(Xf)$ is not a tangent vector at each point p. However, by the commutativity of the partial derivative, the map $f \mapsto X(Yf) - Y(Xf)$ does not contain the second derivative of f, and hence it is a tangent vector at each point p in the sense of Definition 2.7.

Definition 4.1. For vector fields X, $Y \in \mathfrak{X}(M)$, the vector field $[X,Y]$ defined by $[X,Y]f = X(Yf) - Y(Xf)$ is called the *Lie bracket* of X and Y.

The definition yields

Lemma 4.2. *For X, Y, $Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, it hold that*

- $[X,Y] = -[Y,X]$,
- $[fX,Y] = f[X,Y] - (Yf)X$, $[X,fY] = f[X,Y] + (Xf)Y$,
- $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]$.

By (4.2), we obtain

Lemma 4.3. *Under the local expression* (4.1)*, the Lie bracket is expressed as*

$$[X,Y] = \sum_{j,l=1}^n \left(X^j \frac{\partial Y^l}{\partial x^j} - Y^j \frac{\partial X^l}{\partial x^j}\right)\frac{\partial}{\partial x^l}. \tag{4.3}$$

In particular, $[\partial/\partial x^i, \partial/\partial x^j] = \mathbf{0}$.

The Lie bracket is a kind of integrability condition[4]:

Fact 4.4. *Let $[X_1, \dots, X_n]$ be an n-tuple of vector fields on a domain $U \subset M$, which is a basis of T_pM at a point $p \in U$. Then, there exists a local coordinate system $(x^1, \dots, x^n)$ around p satisfying $X_j = \partial/\partial x^j$ $(j = 1, \dots, n)$ if and only if $[X_j, X_k] = \mathbf{0}$ for all $j, k = 1, \dots, n$.*

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[4]The fact will be proven in the lecture on next quarter.

Vector fields on Euclidean space As seen in Example 2.14, a vector field X on $\mathbb{R}^n$ (or $\mathbb{E}^n$, $\mathbb{E}^n_1$, ..., depending on the context) is considered as a smooth map $X\colon \mathbb{R}^n \to \mathbb{R}^n$.

For a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ and a tangent vector $\boldsymbol{v} \in T_p\mathbb{R}^n$, we define the *directional derivative* $D_{\boldsymbol{v}}X$ of X in the direction $\boldsymbol{v}$ as

$$D_{\boldsymbol{v}}X := dX(\boldsymbol{v}) = (d(X^1)(\boldsymbol{v}), \dots, d(X^n)(\boldsymbol{v}))^T, \tag{4.4}$$

where X^1, ..., X^n are the components of X which are smooth functions on $\mathbb{R}^n$. The directional derivative D induces a bilinear correspondence

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto D_X Y \in \mathfrak{X}(M).$$

We call this the *canonical connection* of $\mathbb{R}^n$.

Example 4.5. The correspondence

$$\boldsymbol{x}\colon \mathbb{R}^n \ni \boldsymbol{x} = (x^1, \dots, x^n)^T \mapsto \boldsymbol{x} \in \mathbb{R}^n$$

can be interpreted as a vector field on $\mathbb{R}^n$, which is called the *position vector field*. For any vector field $X \in \mathfrak{X}(\mathbb{R}^n)$,

$$D_X\boldsymbol{x} = X$$

holds. In fact,

$$D_X\boldsymbol{x} = d\boldsymbol{x}(X) = \left.\frac{d}{dt}\right|_{t=0}(\boldsymbol{x} + tX) = X.$$

In the local expression

$$X = (X^1, \dots, X^n)^T = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}, \qquad Y = (Y^1, \dots, Y^n)^T = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}$$

with respect to the canonical coordinate system $(x^1, \dots, x^n)$ of $\mathbb{R}^n$,

$$D_X Y = \sum_{j=1}^n \left(\sum_{i=1}^n X^i \frac{\partial Y^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{4.5}$$

holds.

Lemma 4.6. *The map $(X, Y) \mapsto D_X Y$ is bilinear. Moreover, for X, Y, $Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, it hold that*

(1) $D_{fX}Y = fD_X Y$, $D_X(fY) = fD_X Y + (Xf)Y$,

(2) $D_X Y - D_Y X = [X, Y]$,

(3) $X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$,

where $\langle\ ,\ \rangle$ is the canonical inner product of the Euclidean space $\mathbb{E}^n$ (resp. the Lorentz-Minkowski space $\mathbb{L}^n = \mathbb{E}^n_1$, $\mathbb{E}^n_2$,...).

Proof. Local expressions (4.3), (4.5) and the definition

$$\langle X, Y \rangle = \sum_{j=1}^n (\pm X^j Y^j) \qquad X = (X^1, \dots, X^n)^T \quad \text{and} \quad Y = (Y^1, \dots, Y^n)^T$$

of the inner product yield the conclusion. □

Induced connection on submanifolds of the Euclidean space Let $\mathbb{E}^{n+r}$ be the Euclidean $(n+r)$-space with inner product $\langle\ ,\ \rangle$, and $M \subset \mathbb{E}^{n+r}$ a submanifold of dimension n, where n and r are positive integers. As seen in Example 2.23, a Riemannian metric g on M is obtained by restricting $\langle\ ,\ \rangle$ to the tangent space of M.

Lemma 4.7. *For each point $\boldsymbol{x} \in M$, the orthogonal complement*

$$N_{\boldsymbol{x}} := \bigl(T_{\boldsymbol{x}}M\bigr)^{\perp} = \{\boldsymbol{v} \in \mathbb{E}^{n+r} = T_{\boldsymbol{x}}\mathbb{E}^{n+r}\,;\, \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0 \quad \text{for all} \quad \boldsymbol{w} \in T_{\boldsymbol{x}}M\} \tag{4.6}$$

is an r-dimensional linear subspace of $T_{\boldsymbol{x}}\mathbb{E}^{n+r} = \mathbb{E}^{n+r}$ such that

$$T_{\boldsymbol{x}}\mathbb{E}^{n+r} = \mathbb{E}^{n+r} = T_{\boldsymbol{x}}M \oplus N_{\boldsymbol{x}}. \tag{4.7}$$

Proof. Take an orthonormal basis $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ of $T_{\boldsymbol{x}}M$ and consider a liner map

$$\boldsymbol{\varphi}\colon \mathbb{E}^{n+r} \ni \boldsymbol{v} \mapsto (\langle \boldsymbol{v}, \boldsymbol{e}_j\rangle)_{j=1,\dots,n} \in \mathbb{R}^n.$$

Since $[\boldsymbol{\varphi}(\boldsymbol{e}_i)]_{i=1,\dots,n}$ spans the $\mathbb{R}^n$, $N_{\boldsymbol{x}} = \operatorname{Ker}\boldsymbol{\varphi}$ is an r-dimensional subspace of $\mathbb{E}^{n+r}$. Moreover, $T_{\boldsymbol{x}}M \cap N_{\boldsymbol{x}} = \{\boldsymbol{0}\}$ because $\boldsymbol{v} \in T_{\boldsymbol{x}}M \cap N_{\boldsymbol{x}}$ implies $\langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0$. Hence $\mathbb{E}^{n+r} = T_{\boldsymbol{x}}M \oplus N_{\boldsymbol{x}}$ holds. □

Definition 4.8. The subspace $N_{\boldsymbol{x}}$ in (4.6) is called the *normal space* of M at $\boldsymbol{x}$. For a vector $\boldsymbol{v} \in \mathbb{E}^{n+r}$, $[\boldsymbol{v}]^{\mathrm{T}} \in T_{\boldsymbol{x}}M$ and $[\boldsymbol{v}]^{\mathrm{N}} \in N_{\boldsymbol{x}}$ satisfying

$$\boldsymbol{v} = [\boldsymbol{v}]^{\mathrm{T}} + [\boldsymbol{v}]^{\mathrm{N}}$$

are called the *tangential component* and the *normal component* of $\boldsymbol{v}$, respectively.

Example 4.9. Let $S^n(k) \subset \mathbb{E}^{n+1}$ be the sphere as in Example 2.19, where $k > 0$ is a constant. Since $T_{\boldsymbol{x}}S^n(k) = \boldsymbol{x}^{\perp}$, the normal space $N_{\boldsymbol{x}}$ is the 1-dimensional subspace $\mathbb{R}\boldsymbol{x}$ spanned by $\boldsymbol{x}$. The tangent and normal components of $\boldsymbol{v} \in T_{\boldsymbol{x}} \in \mathbb{E}^{n+1}$ is obtained by

$$[\boldsymbol{v}]^{\mathrm{N}} = \langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e}, \qquad [\boldsymbol{v}]^{\mathrm{T}} = \boldsymbol{v} - \langle \boldsymbol{v}, \boldsymbol{e}\rangle\, \boldsymbol{e} \qquad \Bigl(\boldsymbol{e} := \sqrt{k}\boldsymbol{x}\Bigr).$$

Definition 4.10. Let $M \subset \mathbb{E}^{n+r}$ be an n-dimensional submanifold with Riemannian metric induced by the canonical metric $\langle\ ,\ \rangle$ of $\mathbb{E}^{n+r}$. The map ∇[5]

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y := [D_X Y]^{\mathrm{T}} \in \mathfrak{X}(M)$$

is called the *connection* of the Riemannian manifold $(M, \langle\ ,\ \rangle)$ induced from the canonical connection of $\mathbb{E}^{n+r}$

Remark 4.11. Recall that $D_{\boldsymbol{v}}X$ for $\boldsymbol{v} \in \mathbb{E}^{n+r}$ is the directional derivative of vector-valued function X. So $D_X Y$ is well-defined for vector fields on M. In fact, at a point $p \in M$, take a curve $\gamma(t)$ on M with $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Then $D_X Y(p)$ is defined by $\frac{d}{dt}\big|_{t=0} Y(\gamma(t))$ as a derivative of the vector-valued function Y. In particular, on a local coordinate system $(u^1, \dots, u^n)$ of M, $D_{\partial/\partial u^j} Y = \partial Y/\partial u^j$.

In the situation above, we define a notion of geodesics on the submanifold $M \subset \mathbb{E}^{n+r}$: Let $\gamma(t)$ be a curve on M. Then the *velocity vector field* is the correspondence $\dot\gamma$ defined by

$$t \mapsto \dot\gamma(t) \in T_{\gamma(t)}M. \tag{4.8}$$

Moreover, the *acceleration vector field* $\ddot\gamma$ is defined by

$$t \mapsto \nabla_{\dot\gamma(t)}\dot\gamma(t) \in T_{\gamma(t)}M. \tag{4.9}$$

[5]pronounced "nabla"

Definition 4.12 (Geodesics)**.** A curve $\gamma(t)$ on M whose acceleration vector field vanishes identically is called a *geodesic* on M.

Example 4.13. [Geodiscs of the sphere] Let $S^n(k) \subset \mathbb{E}^{n+1}$ be the sphere of curvature k, where $k > 0$ is a constant. Fix a point $\boldsymbol{x} \in S^n(k)$ and take a unit vector $\boldsymbol{v} \in T_{\boldsymbol{x}} S^n(k) = \boldsymbol{x}^{\perp}$. Set

$$\gamma(t) := \frac{1}{\sqrt{k}} \Big(\cos(\sqrt{k}) t \boldsymbol{e} + \sin(\sqrt{k} t) \boldsymbol{v} \Big) \qquad \big(\boldsymbol{e} := \sqrt{k}\boldsymbol{x}\big).$$

Since $\boldsymbol{e}$ and $\boldsymbol{v}$ are unit vectors which are perpendicular each other, $\langle \gamma, \gamma \rangle = 1/\sqrt{k}$. Hence γ is a curve on $S^n(k)$, and we obtain

$$\dot{\gamma}(t) = -\sin(\sqrt{k}t)\boldsymbol{e} + \cos(\sqrt{k}t)\boldsymbol{v} \in T_{\gamma(t)} S^n(k).$$

Moreover the acceleration vector $\ddot{\gamma}$ of γ as a curve in $\mathbb{E}^{n+1}$ is obtained as

$$\ddot{\gamma}(t) = -\sqrt{k}(\cos(\sqrt{k}t)\boldsymbol{e} + \sin(\sqrt{k}t)\boldsymbol{v}) = -k\gamma(t) \in N_{\gamma(t)}.$$

Hence $\nabla_{\dot{\gamma}} \dot{\gamma} = \boldsymbol{0}$, and the curve is a geodesic on $S^n(k)$.

Exercises

Set

$$H^2(-1)=\{\boldsymbol{x}=(x^0,x^1,x^2)^T\in\mathbb{E}^3_1\,;\,\langle\boldsymbol{x},\boldsymbol{x}\rangle=-1,x_0>0\}.$$

4-1 Let $D:=\{(u,v)\in\mathbb{R}^2\,;\,u^2+v^2<1\}$, and set

$$\boldsymbol{f}:D\ni(u,v)\mapsto\frac{1}{1-u^2-v^2}\bigl(1+u^2+v^2,2u,2v\bigr)\in H^3(-1)$$

and take an orthonormal frame $[\boldsymbol{e}_0(u,v),\boldsymbol{e}_1(u,v),\boldsymbol{e}_2(u,v)]$ as in Problem 3-2.

- Compute the Lie bracket $[\boldsymbol{e}_1,\boldsymbol{e}_2]$ as a liner combination of $\boldsymbol{e}_0$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.
- Compute $D_{\boldsymbol{e}_i}\boldsymbol{e}_j$ for i, $j=1,2$.

4-2

- For each $\boldsymbol{x}\in H^2(-1)$, show that

  $$(*)\qquad\mathbb{E}^3_1=T_{\boldsymbol{x}}H^2(-1)\oplus\mathbb{R}\boldsymbol{x}.$$

- Let $\boldsymbol{x}\in H^2(-1)$ and take a unit vector $\boldsymbol{v}\in T_{\boldsymbol{x}}H^2(-1)=\boldsymbol{x}^\perp$. Then show that

  $$\gamma(t):=(\cosh t)\boldsymbol{x}+(\sinh t)\boldsymbol{v}$$

  is a curve on $H^2(-1)$ satisfying $[\ddot\gamma(t)]^{\mathrm{T}}=\mathbf{0}$, where $[*]^{\mathrm{T}}$ denotes the $T_{\gamma(t)}H^2(-1)$-components of the decomposition $(*)$ with $\boldsymbol{x}=\gamma(t)$.