5 Riemannian connection

Riemannian connection Let (M, g) be a (pseudo) Riemannian manifold, and denote by $\langle\ ,\ \rangle$ the inner product induced by g. We let $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ be the set of smooth functions and smooth vector fields on M, respectively.

Lemma 5.1. *There exists the unique bilinear map*

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \longmapsto \nabla_X Y \in \mathfrak{X}(M)$$

satisfying

- $\nabla_X Y - \nabla_Y X = [X, Y]$,
- $X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

for X, Y, $Z \in \mathfrak{X}(M)$, *where* $[\ ,\ ]$ *denotes the Lie bracket in Definition 4.1.*

Proof. Set

$$C(X,Y,Z) := \frac{1}{2}\bigl(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle + \langle X,[Z,Y]\rangle - \langle Z,[Y,X]\rangle\bigr). \tag{5.1}$$

Then there exists unique vector field $\nabla_X Y$ satisfying

$$\langle \nabla_X Y, Z\rangle = C(X, Y, Z).$$

This is the desired one. □

Definition 5.2. The map ∇ in Lemma 5.1 is called the *Riemannian connection* or the *Levi-Civita connection*.

Lemma 5.3. *The Levi-Civita connection satisfies*

- $\nabla_{fX} Y = f\nabla_X Y$,
- $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$

for all X, $Y \in \mathfrak{X}(M)$ *and* $f \in \mathcal{F}(M)$.

Proof. Lemma 4.2 and the equation (5.1) yields the conclusion. □

Corollary 5.4. *Assume* X, $Y \in \mathfrak{X}(M)$ *satisfy* $X_p = Y_p$ *at a point* $p \in M$. *Then*

$$(\nabla_X Z)_p = (\nabla_Y Z)_p$$

holds for each $Z \in \mathfrak{X}(M)$.

Proof. Take an $n = \dim M$-tuple of vector fields $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on a domain $U \subset M$ which gives a basis of T_pM for each point $p \in U$. Then a vector field X is expressed as

$$X = \sum_{j=1}^{n} X^j \boldsymbol{e}_j \qquad \text{and} \qquad Y = \sum_{j=1}^{n} Y^j \boldsymbol{e}_j$$

where X^j and Y^j are smooth functions satisfying $X^j(p) = Y^j(p)$ $(j = 1, \dots, n)$. Then by the first assertion of Lemma 5.3, we have

$$(\nabla_X Z)_p = \sum_{j=1}^{n} X^j(p)(\nabla_{\boldsymbol{e}_j} Z)_p = \sum_{j=1}^{n} Y^j(p)(\nabla_{\boldsymbol{e}_j} Z)_p = (\nabla_Y Z)_p. \qquad \square$$

Definition 5.5. For $\boldsymbol{x} \in T_pM$ be a tangent vector at $p \in M$ and a vector field $Y \in \mathfrak{X}(M)$,

$$\nabla_{\boldsymbol{x}} Y := (\nabla_X Y)_p \in T_pM$$

is called the *covariant derivative* of Y with respect to the direction $\boldsymbol{x}$, where $X \in \mathfrak{X}(M)$ is a vector field satisfying $X_p = \boldsymbol{x}$.

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Examples

Example 5.6. Let $\mathbb{E}^{n+r}_r$ be the pseudo Euclidean space and $\langle\ ,\ \rangle$ the inner product of signature (n,r). Then the the canonical connection (cf. (4.4)) D defined by

$$D\colon \mathfrak{X}(\mathbb{E}^{n+r}_r)\times\mathfrak{X}(\mathbb{E}^{n+r}_r)\ni(X,Y)\longmapsto D_XY:=dY(X)\in\mathfrak{X}(M)$$

c is the Levi-Civita connection, because of Lemma 4.6.

Lemma 5.7. *Let $M\subset\mathbb{E}^{n+r}_r$ be a submanifold of the pseudo Euclidean space $\mathbb{E}^{n+r}_r$. If the restriction of the inner product $\langle\ ,\ \rangle$ of $\mathbb{E}^{n+r}_r$ on T_pM is non-degenerate, the direct sum decomposition*

$$\mathbb{E}^{n+r}_r=T_pM\oplus(T_pM)^\perp,\qquad (T_pM)^\perp:=\{\boldsymbol{v}\in\mathbb{E}^{n+r}_r\,;\,\langle\boldsymbol{x},\boldsymbol{v}\rangle=0 \text{ for all } \boldsymbol{x}\in T_pM\},$$

that is, for each vector $\boldsymbol{v}\in\mathbb{E}^{n+r}_r=T_p\mathbb{E}^{n+r}_r$, there exists a unique decomposition

$$\boldsymbol{v}=[\boldsymbol{v}]^{\mathrm{T}}+[\boldsymbol{v}]^{\mathrm{N}},\qquad [\boldsymbol{v}]^{\mathrm{T}}\in T_pM,\qquad [\boldsymbol{v}]^{\mathrm{N}}\in(T_pM)^\perp. \tag{5.2}$$

The vectors $[\boldsymbol{v}]^{\mathrm{T}}$ and $[\boldsymbol{v}]^{\mathrm{N}}$ in (5.2) are called the *tangential component* and *normal component* of $\boldsymbol{v}$, respectively.

Proof. By the relationship of rank and kernel, $\dim T_pM+\dim(T_pM)^\perp=\dim\mathbb{E}^{n+r}_r$. Let $\boldsymbol{v}\in T_pM\cap(T_pM)^\perp$. Since $\boldsymbol{v}$ in $(T_pM)^\perp$, $\langle\boldsymbol{v},\boldsymbol{w}\rangle=0$ for all $\boldsymbol{w}\in T_pM$. Here $\boldsymbol{v}\in T_pM$ and $\langle\ ,\ \rangle|_{T_pM}$ is non-degenerate, $\boldsymbol{v}=\boldsymbol{0}$. Hence the conclusion follows. □

Remark 5.8. When $r=0$, that is, the case that $\mathbb{E}^n$ is the Euclidean space, the non-degeneracy assumption of Lemma 5.7 is empty. On the other hand, in $\mathbb{E}^3_1$, for example,

$$M:=\{(u,u,v)\,;\,u,v\in\mathbb{R}\}$$

does not satisfy the non-degeneracy assumption. In fact, $T_pM=\operatorname{Span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}$, where $\boldsymbol{v}_1=(1,1,0)^T$ and $\boldsymbol{v}_2=(0,0,1)^T$. Then $\langle\boldsymbol{v}_1,\boldsymbol{x}\rangle=0$ for all $\boldsymbol{x}\in T_pM$, that is, $\langle\ ,\ \rangle|_{T_pM}$ degenerates. In this case, $(T_pM)^\perp=\mathbb{R}\boldsymbol{v}_1\subset T_pM$.

Theorem 5.9. *Let $M\subset\mathbb{E}^{n+r}_r$ be a submanifold such that the restriction of the inner product $\langle\ ,\ \rangle$ to TM is non-degenerate. We set ∇_XY for X, $Y\in\mathfrak{X}(M)$ by*

$$\nabla_XY:=[D_XY]^{\mathrm{T}}.$$

Then ∇ is the Levi-Civita connection of M with respect to the induced metric $\langle\ ,\ \rangle|_{TM}$.

Proof. For X and $Y\in\mathfrak{X}(M)$, $[X,Y]\in\mathfrak{X}(M)$ holds. Hence

$$\nabla_XY-\nabla_YX=[D_XY-D_YX]^{\mathrm{T}}=[[X,Y]]^{\mathrm{T}}=[X,Y]$$

yield the first assertion of Lemma 5.1. On the other hand,

$$X\langle Y,Z\rangle=\langle D_XY,Z\rangle+\langle Y,D_XZ\rangle=\left\langle[D_XY]^{\mathrm{T}},Z\right\rangle+\left\langle Y,[D_XZ]^{\mathrm{T}}\right\rangle=\langle\nabla_XY,Z\rangle+\langle Y,\nabla_XZ\rangle.$$

Hence ∇ is the Levi-Civita connection. □

Exercises

Set

$$H^2(-1) = \{\boldsymbol{x} = (x^0, x^1, x^2)^T \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = -1, x^0 > 0\},$$

and take a parametrization

$$\boldsymbol{f} : D \ni (u, v) \mapsto \frac{1}{1 - u^2 - v^2}\bigl(1 + u^2 + v^2, 2u, 2v\bigr) \in H^2(-1)$$

of $H^2(-1)$, where $D := \{(u, v) \in \mathbb{R}^2\,;\, u^2 + v^2 < 1\}$.

5-1 Let $[\boldsymbol{e}_0(u, v), \boldsymbol{e}_1(u, v), \boldsymbol{e}_2(u, v)]$ be an orthonormal frame defined by

$$\boldsymbol{e}_0 := \boldsymbol{f}, \qquad \boldsymbol{e}_1 := \frac{\boldsymbol{f}_u}{|\boldsymbol{f}_u|}, \qquad \boldsymbol{e}_2 := \frac{\boldsymbol{f}_v}{|\boldsymbol{f}_v|},$$

as in Problem 4-1. For the induced connection ∇ of $H^2(-1)$,

- Compute $\langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k\rangle$ for i, j and k run over $\{1, 2\}$.
- Compute
$$\nabla_{\boldsymbol{e}_1}\nabla_{\boldsymbol{e}_2}\boldsymbol{e}_2 - \nabla_{\boldsymbol{e}_2}\nabla_{\boldsymbol{e}_1}\boldsymbol{e}_2 - \nabla_{[\boldsymbol{e}_1,\boldsymbol{e}_2]}\boldsymbol{e}_2.$$

5-2 Let $\widetilde{D} := (0, \infty) \times (-\pi, \pi)$ and take another parametrization

$$\widetilde{\boldsymbol{f}} : \widetilde{D} \ni (r, t) \mapsto (\cosh r, \sinh r \cos t, \sinh r \sin t)^T \in H^2(-1)$$

of $H^2(-1)$, and set

$$\boldsymbol{v}_0 = \widetilde{\boldsymbol{f}}, \qquad \boldsymbol{v}_1 = \widetilde{\boldsymbol{f}}_r, \qquad \boldsymbol{v}_2 = \frac{1}{\sinh r}\widetilde{\boldsymbol{f}}_t.$$

- Find a parameter change $\varphi\colon (r, t) \mapsto (u, v) = \bigl(u(r, t), v(r, t)\bigr)$.
- Find a 2×2-matrix valued function $\Theta = \Theta(r, t)$ satisfying
$$[\boldsymbol{e}_1, \boldsymbol{e}_2] = [\boldsymbol{v}_1, \boldsymbol{v}_2]\Theta,$$
where the left-hand side and the right-hand side are valuated at $(u(r, t), v(r, t))$ and (r, t), respectively.