6 Geodesics

Let (M, g) be an n-dimensional pseudo Riemannian manifold, and denote by ∇ the Levi-Civita connection.

Pregeodesics and geodesics For a smooth curve $\gamma\colon I \to M$ defined on an interval $I \subset \mathbb{R}$, the velocity $\dot\gamma$ and the acceleration $\nabla_{\dot\gamma}\dot\gamma$ are defined.

Definition 6.1. A curve $\gamma = \gamma(t)$ is called a *pregeodesic* if $\nabla_{\dot\gamma}\dot\gamma$ is parallel to $\dot\gamma$, that is, there exists a smooth function $\varphi(t)$ in t such that $\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma$.

Remark 6.2. A notion of pregeodesic does not depend on a choice of parameter of the curve. In fact, let $\gamma(t)$ be a curve on M, and $t = t(s)$ a parameter change, that is a smooth function in s with $dt/ds > 0$ everywhere. Then the parameter change $\tilde\gamma(s) := \gamma(t(s))$ of γ satisfies

$$\begin{aligned} \gamma' &:= \frac{d\tilde\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt} = \frac{dt}{ds}\dot\gamma, \\ \nabla_{\gamma'}\gamma' &= \nabla_{\gamma'}\frac{dt}{ds}\dot\gamma = \frac{d^2t}{ds^2}\dot\gamma + \frac{dt}{ds}\nabla_{\frac{dt}{ds}\dot\gamma}\dot\gamma = \frac{d^2t}{ds^2}\dot\gamma + \left(\frac{dt}{ds}\right)^2\nabla_{\dot\gamma}\dot\gamma. \end{aligned} \tag{6.1}$$

Hence $\nabla_{\gamma'}\gamma'$ is proportional to γ' if and only if $\nabla_{\dot\gamma}\dot\gamma$ is proportional to $\dot\gamma$.

Definition 6.3. A curve γ is called a *geodesic* if $\nabla_{\dot\gamma}\dot\gamma = \mathbf{0}$ holds identically.

Lemma 6.4. *If γ is a geodesic, then $\langle\dot\gamma, \dot\gamma\rangle$ is constant.*

Proof. By the definition of the Levi-Civita connection (Definition 5.2),

$$\frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0.$$

Hence $\langle\dot\gamma, \dot\gamma\rangle = 0$. □

Remark 6.5. By virtue of Lemma 6.4, the notion of geodesics *does* depend on parameters, unlike the pregeodesics.

By definition, a geodesic is a pregeodesic. Though the converse is not true in general, a pregeodesic coincides a geodesic up to a parameter change.

Lemma 6.6. *Let $\gamma\colon I \ni t \mapsto \gamma(t) \in M$ be a geodesic, where $I \subset \mathbb{R}$ is an interval. Then there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) = \gamma(t(s))$ is a geodesic.*

Proof. Take a function $\varphi\colon I \to \mathbb{R}$ such that $\nabla_{\dot\gamma}\dot\gamma = \varphi\dot\gamma$. We define a function $s\colon I \to \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t}\left(\exp\int_{t_0}^{u}\varphi(\tau)\,d\tau\right)du,$$

where $t_0 \in I$ is an arbitrarily fixed point. Since $ds/dt > 0$ holds everywhere, $s\colon I \mapsto I' := s(I)$ is a diffeomorphism and the inverse $t = t(s)$ exists. Since

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \exp\left(-\int_{t_0}^{t}\varphi(u)\,du\right),$$

(6.1) yields $\nabla_{\gamma'}\gamma' = \mathbf{0}$, where $' = d/ds$. □

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Example 6.7. Let $M \subset \mathbb{E}^3$ be a 2-dimensional submanifold of the Euclidean space, and take the unit normal vector field ν along M. Since the tangent space T_pM is the orthogonal complement of $\nu(p)$ for all $p \in M$,

$$\nabla_{\dot\gamma}\dot\gamma = [\ddot\gamma]^{\mathrm{T}} = \ddot\gamma - \langle \ddot\gamma, \nu\rangle\, \nu$$

holds for a curve γ on M. Then the curve γ is a pregeodesic if and only if $\ddot\gamma$ is linearly dependent to $\{\dot\gamma, \nu\}$, that is,

$$\det(\dot\gamma, \ddot\gamma, \hat\nu) = 0$$

holds, where $\hat\nu(t) = \nu \circ \gamma(t)$ is the unit normal vector field of the surface M along the curve γ.

Existence and Uniqueness

Fact 6.8. *For each $p \in M$ and $\boldsymbol{v} \in T_pM$, there exists unique geodesic $\gamma_{p,\boldsymbol{v}} \colon I \to M$, where I is an interval including 0 such that $\gamma(0) = p$ and $\dot\gamma(0) = \boldsymbol{v}$.*

Remark 6.9. Fact 6.8 can be proven by the fundamental theorem for ordinary differential equations, because the equation $\nabla_{\dot\gamma}\dot\gamma = 0$ is a system of ordinary differential equation of the coordinate functions of $\gamma(t)$ on a coordinate neighborhood. A brief review of theory of ordinary differential equations will be given in lectures on next quarter.

For each $p \in M$ and $\boldsymbol{v} \in T_pM$, we denote by $\gamma_{p,\boldsymbol{v}}$ the geodesic with

$$\gamma_{p,\boldsymbol{v}}(0) = p, \qquad \dot\gamma_{p,\boldsymbol{v}}(0) = \boldsymbol{v}.$$

Proposition 6.10. *For arbitrary constant k, $\gamma_{p,k\boldsymbol{v}}(t) = \gamma_{p,\boldsymbol{v}}(kt)$ holds.*

Proof. Let $\gamma(t) = \gamma_{p,\boldsymbol{v}}(kt)$. Then $\dot\gamma(t) = k\dot\gamma_{p,\boldsymbol{v}}(kt)$, and $\nabla_{\dot\gamma}\dot\gamma = k^2\nabla_{\dot\gamma_{p,\boldsymbol{v}}}\gamma_{p,\boldsymbol{v}}$. Hence $\gamma(t)$ is a geodesic. Moreover, by definition, $\gamma(0) = p$ and $\dot\gamma(0) = k\boldsymbol{v}$. Hence $\gamma_{p,k\boldsymbol{v}} = \gamma$ by the uniqueness. $\square$

Example 6.11. Let $k > 0$ be a constant and

$$S^n(k) := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1}\,;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{k} \right\}$$

be the n-dimensional sphere of curvature k. Since for each $\boldsymbol{x} \in S^n(k)$, $T_{\boldsymbol{x}}S^n(k) = \boldsymbol{x}^\perp$ holds. For given $\boldsymbol{x} \in S^n(k)$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}S^n(k) \setminus \{\boldsymbol{0}\}$, we set

$$\gamma(t) := (\cos\sqrt{k}vt)\boldsymbol{x} + (\sin\sqrt{k}vt)\boldsymbol{v}' \qquad \left(v = \langle \boldsymbol{v}, \boldsymbol{v}\rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{\sqrt{k}v} \right).$$

Since $\ddot\gamma(t)$ is proportional to $\gamma(t)$, $\nabla_{\dot\gamma}\dot\gamma = \boldsymbol{0}$. Hence γ is a geodesic with $\gamma(0) = \boldsymbol{x}$, $\dot\gamma(0) = \boldsymbol{v}$.

Completeness

Definition 6.12. A pseudo Riemannian manifold (M, g) is said to be *complete* if all geodesics are defined on whole on $\mathbb{R}$.

Properties of complete *Riemannian* manifolds will be treated in the next lecture.

Exercises

6-1 Let

$$\boldsymbol{f} : D = (0,\infty)\times(-\pi,\pi) \ni (r,t) \mapsto (\cosh r, \sinh r\cos t, \sinh r \sin t)^T \in H^2(-1)$$

be a parametrization in $H^2(-1)$ as in Problem 5-2. Show that $\gamma(r)\colon r \mapsto \boldsymbol{f}(r,t) \in H^2(-1)$ is a geodesic for each fixed value t.

6-2 Let

$$S^2_1 := \{\boldsymbol{x} \in \mathbb{E}^3_1\,;\, \langle \boldsymbol{x},\boldsymbol{x}\rangle = 1\},$$

which is called the *de Sitter plane*. Then the restriction of the inner product of the Lorentz-Minkowski space $\mathbb{E}^3_1$ to the tangent space $T_{\boldsymbol{x}}S^3_1 = \boldsymbol{x}^\perp$ is of sign $(1,1)$, that is, S^2_1 is a Lorentzian manifold. For each $\boldsymbol{x} \in S^2_1$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}S^2_1 \setminus \{0\}$, we set

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cosh vt)\boldsymbol{x} + (\sinh vt)\boldsymbol{v'} & \text{if } \langle \boldsymbol{v},\boldsymbol{v}\rangle < 0,\\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle \boldsymbol{v},\boldsymbol{v}\rangle = 0,\\ (\cos vt)\boldsymbol{x} + (\sin vt)\boldsymbol{v'} & \text{if } \langle \boldsymbol{v},\boldsymbol{v}\rangle > 0,\end{cases}$$

where $v := |\langle \boldsymbol{v},\boldsymbol{v}\rangle|^{1/2}$ and $\boldsymbol{v'} := \boldsymbol{v}/v$. Show that $\gamma := \gamma_{\boldsymbol{x},\boldsymbol{v}}$ is a geodesic on S^3_1 with $\gamma(0) = \boldsymbol{x}$ and $\dot\gamma(0) = \boldsymbol{v}$.