## 7 Hopf-Rinow's theorem

In this section, we assume $(M, g)$ is an $n$-dimensional connected *Riemannian* manifold[6].

### Distance

**Definition 7.1.** Let $\gamma\colon [a, b] \to M$ be a piecewize $C^1$-curve, where $[a, b]$ is a closed interval on $\mathbb{R}$. The integral

$$\mathcal{L}(\gamma) := \int_a^b |\dot{\gamma}(t)|\, dt$$

is called the *length* of $\gamma$.

**Lemma 7.2.** *For points $p$, $q \in M$, we denote*

$$\mathcal{C}_{p,q} := \left\{\gamma\colon [a, b] \to M\, ;\, \gamma \text{ is a piecewize } C^1\text{-curve with } \gamma(a) = p \text{ and } \gamma(b) = q\right\}.$$

*Then*

$$d(p, q) := \inf\{\mathcal{L}(\gamma)\, ;\, \gamma \in \mathcal{C}_{p,q}\}\colon M \times M \to \mathbb{R}$$

*is a distance function on $M$, that is, it satisfies the axiom*

- $d(p, q) \geqq 0$ *for any $p$, $q \in M$. The equality holds if and only if $p = q$.*
- $d(p, q) = d(q, p)$.
- $d(p, q) + d(q, r) \geqq d(q, r)$

*of distance. Moreover, the topology of $M$ induced by the distance $d$ coincides with the original topology of $M$.*

**Definition 7.3.** The distance function $d$ in Lemma 7.2 is called the *distance induced from the Riemannian metric $g$*, or the *Riemannian distance with respect to $g$*.

**Proposition 7.4.** *For $p$, $q \in M$, a curve $\gamma \in \mathcal{C}_{p,q}$ satisfying $d(p, q) = \mathcal{L}(\gamma)$ is a pregeodesic.*

**Definition 7.5.** The geodesic $\gamma \in \mathcal{C}_{p,q}$ satisfying $\mathcal{L}(\gamma) = d(p, q)$ is called the *minimizing geodesic* or the *shortest geodesic* joining $p$ and $q$.

**Completeness** We denote by $\gamma_{p,\boldsymbol{v}}$ the geodesic on a (pseudo) Riemannian manifold $(M, g)$ starting $p \in M$ with initial velocity $\boldsymbol{v} \in T_pM$. A geodesic $\gamma_{p,\boldsymbol{v}}$ is said to be *complete* if it is defined on whole $\mathbb{R}$. The manifold $(M, g)$ is said to be *complete* if all geodesics are complete.

**Example 7.6.** The Euclidean space $\mathbb{E}^n$ is complete. In fact, $\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v}$ is defined on $\mathbb{R}$.

**Example 7.7.** Let $k > 0$ be a constant and

$$S^n(k) := \left\{\boldsymbol{x} \in \mathbb{E}^{n+1}\, ;\, \langle \boldsymbol{x}, \boldsymbol{x}\rangle = \frac{1}{k}\right\}$$

be the $n$-dimensional sphere of curvature $k$. Then for each $\boldsymbol{x} \in S^n(k)$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}S^n(k)$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cos\sqrt{k}vt)\boldsymbol{x} + (\sin\sqrt{k}vt)\boldsymbol{v}' \qquad \left(v = \langle \boldsymbol{v}, \boldsymbol{v}\rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{\sqrt{k}v}\right)$$

is the geodesic defined on $\mathbb{R}$. Hence $S^n(k)$ is complete.

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[6]The following properties cannot hold for pseudo Riemannian manifolds in general.

**Example 7.8.** Let $k < 0$ be a constant and

$$H^n(k) := \left\{ \boldsymbol{x} = (x^0, \dots, x^{n+1}) \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{k}, x^0 > 0 \right\}$$

be the $n$-dimensional hyperbolic space of curvature $k$. Then for each $\boldsymbol{x} \in H^n(k)$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}H^n(k)$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := (\cosh \sqrt{-k}vt)\boldsymbol{x} + (\sinh \sqrt{-k}vt)\boldsymbol{v}' \qquad \left( v = \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{1/2}, \quad \boldsymbol{v}' := \frac{\boldsymbol{v}}{\sqrt{k}v} \right)$$

is the geodesic defined on $\mathbb{R}$. Hence $H^n(k)$ is complete.

**Example 7.9.** Let $k > 0$ be a constant and

$$S^n_1(k) := \left\{ \boldsymbol{x} \in \mathbb{E}^{n+1}_1 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{k} \right\}$$

be the $n$-dimensional de Sitter space of curvature $k$, which is a Lorentzian manifold. Then for each $\boldsymbol{x} \in S^n_1(k)$ and $\boldsymbol{v} \in T_{\boldsymbol{x}}S^n_1(k)$,

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) := \begin{cases} (\cos \sqrt{k}vt)\boldsymbol{x} + (\sin \sqrt{k}vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle > 0, \\ \boldsymbol{x} + t\boldsymbol{v} & \text{if } \langle v, v \rangle = 0, \\ (\cosh \sqrt{k}vt)\boldsymbol{x} + (\sinh \sqrt{k}vt)\boldsymbol{v}' & \text{if } \langle v, v \rangle < 0, \end{cases}$$

where $v := |\langle \boldsymbol{v}, \boldsymbol{v} \rangle|^{1/2}$, $\boldsymbol{v}' := \frac{\boldsymbol{v}}{\sqrt{k}v}$, is the geodesic defined on $\mathbb{R}$. Hence $S^n_1(k)$ is complete.

**Example 7.10.** The open submanifold $M := \mathbb{E}^n \setminus \{\boldsymbol{0}\}$ of the Euclidean space $\mathbb{E}^n$ is not complete. In fact, let $\boldsymbol{x} \in M$ and $\boldsymbol{v} := -\boldsymbol{x} \in \mathbb{E}^n = T_{\boldsymbol{x}}M$. Then the geodesic

$$\gamma_{\boldsymbol{x},\boldsymbol{v}}(t) = \boldsymbol{x} + t\boldsymbol{v} = (1-t)\boldsymbol{x}$$

is defined only on $(-\infty, 1)$.

### Hopf-Rinow's theorem

**Theorem 7.11** (Hopf-Rinow's theorem)**.** *For a given connected Riemannian manifold $(M, g)$, the following are equivalent:*

(1) *$(M, g)$ is complete.*

(2) *There exists $p \in M$ such that all geodesics emanating at $p$ are complete.*

(3) *$(M, d)$ is a complete distance (metric) space, where $d$ is the distance induced from $g$.*

(4) *Any bounded subset $D$ of $M$ is precompact, that is, the closure of $D$ is compact.*

(5) *Any divergent path has infinite length.*

*Here, a curve $\gamma\colon [0, a) \to M$ is called* divergent path *if for all compact set $K \subset M$, there exists $T > 0$ such that $\gamma([T, a)) \subset M \setminus K$.*

Moreover,

**Theorem 7.12.** *Assume $(M, g)$ is a complete connected Riemannian manifold. Then for any $p$ and $q \in M$, there exists the shortest geodesic joining $p$ and $q$.*