Advanced Topics in Geometry F1 (MTH.B506)

Linear Ordinary Differential Equations

(線型) 常微分方程式

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学

unknown

initial condition

(normal form)
正規形

$$\boxed{\frac{d}{dt}x(t) = \underbrace{f(t, x(t))}_{\text{known}},} \quad \boxed{x(t_0) = x_0} \quad (*)$$

initial value prob.
(IVP)

- Existence
- Uniqueness
- Regularity on initial conditions and parameters

- x : $\mathbb{R}^n$-valued unknown.

  $f(t, x)$ of class C^∞ ← given

  1-var. n-variable

- $\exists 1$ $x : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathbb{R}^n$ satisfying $(*)$

  suff. small

Example

x: $\mathbb{R}^1$-valued.

λ: const.

$$\frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t), \qquad x(0) = x_0.$$

linear in unknowns.

- $\underline{x(t) = x_0 \exp \lambda t}$: sol

defined on $\mathbb{R}$

linear

Example

ω: const

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos \omega t + \frac{y_0}{\omega} \sin \omega t \\ -x_0 \omega \sin \omega t + y_0 \cos \omega t \end{pmatrix}$$

defined on $\mathbb{R}$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

linear

$\dot{x}(t)$

Initial cond.

$$\begin{cases} x(0) = x_0 \\ y(0) = \frac{dx}{dt}(0) = y_0 \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = y. \\ \frac{dy}{dt} = -\omega^2 x \end{cases}$$

(simuletanious eq)

harmonic oscillation

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x .$$

Example

non linear

$$\frac{dx}{dt} = 1 + x^2, \qquad x(0) = 0.$$

$x(t) = \tan t$ $\quad \left(\int_0^{x(t)} \frac{dx}{1+x^2} = \int_0^t dt \right)$

defined only on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

cannot extend beyond $\pm\frac{\pi}{2}$

Example

SIR-model ← a classical (early 20c) model of epidemic

infection rate

recovery rate

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI \\[2ex] \dfrac{dI}{dt} = \beta SI - \gamma I \\[2ex] \dfrac{dR}{dt} = \gamma I \end{cases}$$

t: time

non-linear.

S: susceptible

I: infected

R: recovered

$$\frac{d}{dt}(\underbrace{S+I+R}_{\text{const (total population)}}) = 0$$