Advanced Topics in Geometry F1 (MTH.B506)

Linear Ordinary Differential Equations

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Linear ordinary differential equations

線形

x: unkown $\mathbb{R}^n$-valued

$$\frac{d}{dt}\boldsymbol{x}(t) = \underbrace{A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)}_{\text{(linear in } x)},$$

A: $n\times n$ matrix valued

b: $\mathbb{R}^n$-valued

given

defined on $I \subset \mathbb{R}$

- $\boxed{\text{Global Existence}}$

$(x(t_0) = x_0)$ (IVP)

Thm $\exists^1 x(t)$: the sol. of (IVP) defined on I

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0,$$

$t_0 \in$

X: unknown, $n \times n$-matrix-valued.

Ω, B: matrix-valued data, C^∞, defined on I

Proposition (Prop. 1.8) $(B = 0)$ homogenious

Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Then

$$\det X(t) = (\det X_0) \exp \underbrace{\int_{t_0}^{t} \operatorname{tr} \Omega(\tau)\, d\tau}.$$

In particular, if $X_0 \in \boxed{\mathrm{GL}(n, \mathbb{R})}$*, then* $\boxed{X(t) \in \mathrm{GL}(n, \mathbb{R})}$ *for all* t.

n×n nonsingular matrices
(regular)

$$\frac{dX}{dt} = X\Omega \quad ; \quad \frac{d}{dt}\det X = \operatorname{trace} \tilde{X}\frac{dX}{dt}$$

$$= \operatorname{trace}(\tilde{X}X)\Omega$$

$$= \operatorname{trace}(\det X)\cdot \mathrm{id}\,\Omega$$

$$= (\det X)\operatorname{trace}\Omega$$

$$\det X = (\det X_0)\exp\int_{t_0}^{t}(\operatorname{trace}\Omega(\tau))\,d\tau$$

$$\det X = \operatorname{trace}\tilde{X}\frac{dX}{dt}$$

余因子

$\tilde{X}$: the cofactor matrix

$$\tilde{X}X = X\tilde{X} = (\det X)\,\mathrm{id}$$

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Corollary (Cor. 1.9)

If $\boxed{\operatorname{tr}\Omega(t) = 0,}$ *then* $\boxed{\det X(t) \text{ is constant.}}$ *In particular, if* $X_0 \in \underline{\mathrm{SL}(n,\mathbb{R})}$, X *is a function valued in* $\mathrm{SL}(n,\mathbb{R})$.

subgroup

$\{X \in M_n(\mathbb{R}) ; \det X = 1\} \subset GL(n.\mathbb{R})$

special linear group

$n\times n$ matrices

(lie algebra)

$\mathrm{Lie}(SL(n.\mathbb{R})) = \{ n\times n \text{ matrix with } \mathrm{tr}=0\}$

Preliminaries

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Proposition (Prop. 1.10)

Assume $\boxed{\Omega^T + \Omega = O.}$ skew-symmetric

If $X_0 \in \mathrm{O}(n)$ *(resp.* $X_0 \in \mathrm{SO}(n)$*),*

then $X(t) \in \mathrm{O}(n)$ *(resp.* $X(t) \in \mathrm{SO}(n)$*) for all* t.

orthogonal

$\cdot\ \mathrm{O}(n) = \{ X \in GL(n,\mathbb{R}) ;\ \cancel{X^T X} = X X^T = \mathrm{id} \}$

det $= \pm 1$

$\cdot\ \mathrm{SO}(n) = \mathrm{O}(n) \cap SL(n,\mathbb{R})$

$$\frac{d}{dt}(XX^T) = \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T = X\Omega X^T + X(X\Omega)^T$$

omit

$$= X\Omega X^T + X\Omega^T X^T = X(\Omega + \Omega^T)X^T = O$$

Linear ordinary differential equations.

Proposition (Prop. 1.12)

Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,.$$

Linear ordinary differential equations.

Corollary (Cor. 1.13)

There exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \big(X(t) := X_{t_0,X_0}(t)\big)$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Non-homogenious case

Proposition (Prop. 1.14)

Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0.$$

parameter sp.

$\ni \alpha$

Theorem (Thm. 1.15)

Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t, \boldsymbol{\alpha})$ and $B(t, \boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ $(\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m))$. Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0, X_0, \boldsymbol{\alpha}}(t)$ defined on I such that

m-parameter

$$\frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \qquad X(t_0) = X_0. \qquad (1)$$

family of ODE's

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

C^∞ in $\begin{cases} \text{initial cond } (t_0\ X_0) \\ \text{parameter } \alpha \end{cases}$

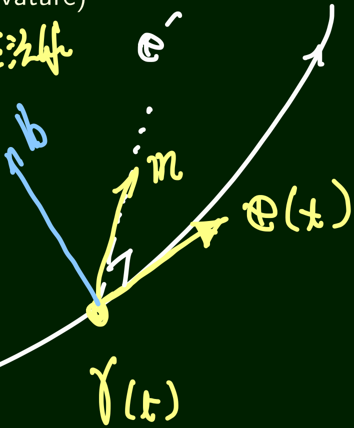
Application to Space Curves 空間曲線の基本定理

- $\gamma\colon I \to \mathbb{R}^3$: a space curve parametrized by the arclength. $|\gamma'|=1$
- $e = \gamma'$
- $\kappa = |e'|$; we assume $\boxed{\kappa > 0}$ (the curvature)
- $n = e'/\kappa$ (the principal normal) 主法線
- $b = e \times n$ (the binormal)
- $\tau = -b' \cdot n$ (the torsion) 3×3

$$F(t) = \begin{pmatrix} e(t) & n(t) & b(t) \end{pmatrix}$$

$$: I \longrightarrow SO(3)$$

Frenet frame (枠)



Frenet-Serret

s: arclength parameter

- $\mathcal{F} := (\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b})\colon I \to \mathrm{SO}(3)$: the Frenet Frame

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

skew symm

$$\frac{de}{ds} = \kappa n$$

$$\frac{dn}{ds} = -\kappa e + 0 n + \tau b$$

$$\left\langle \frac{dn}{ds}, n \right\rangle = \frac{1}{2}\frac{d}{ds}\langle n, n\rangle = 0$$

$$\left\langle \frac{dn}{ds}, e \right\rangle = \frac{d}{ds}\langle n, e\rangle - \left\langle n, \frac{de}{ds} \right\rangle$$

κn

The Fundamental Theorem for Space Curves

Theorem (Thm. 1.17)

Let $\kappa(s)$ and $\tau(s)$ be C^∞-fnctions defined on an interval I satisfying $\kappa(s) > 0$ on I. given

Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation
$\boldsymbol{x} \mapsto A\boldsymbol{x} + \cancel{\boldsymbol{b}}$ $(A \in \mathrm{SO}(3),\ \cancel{\boldsymbol{b}} \in \mathbb{R}^3)$ *of* $\mathbb{R}^3$.

c c

Solve

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \quad \mathcal{F}(s_0) = \mathrm{id}. \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

$\Rightarrow \mathcal{F} : I \longrightarrow SO(3)$

$\Rightarrow$ decompose $\mathcal{F} = (\mathbb{e} \;\; \mathbb{n} \;\; \mathbb{b})$ in column vectors

$\rightarrow$ $\gamma(s) = \displaystyle\int_{s_0}^{s} \mathbb{e}(u)\,du$ is the desired one.

Logistic eq.

Problem (Ex. 1-1)

Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where b *is a real number.*

a

Exercise 1-2

Problem (Ex. 1-2)

Find an explicit expression of a space curve $\gamma(s)$ *parametrized by the arc-length* s*, whose curvature* κ *and torsion* τ *satisfy*

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

parameter change

const.

$$\frac{d\mathcal{F}}{ds} = \frac{1}{1+s^2}\mathcal{F}\Omega_0$$

$$\frac{d\mathcal{F}}{du} = \mathcal{F}\Omega_0 \qquad \mathcal{F} = \exp u\Omega_0$$