Advanced Topics in Geometry F1 (MTH.B506)

Integrablity Conditions

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-f1/

Tokyo Institute of Technology

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Problem 1-1

non extendable

Problem (Ex. 1-1)

Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

the logistic equation

· population growth ....

$$\frac{dx}{dt} = x(1-x) \qquad x(0) = a$$

- When $a = 0, 1 \Rightarrow x(t) = a$ (const) is the unique sol
- When $a \notin \{0, 1\} \Rightarrow x(t) \notin \{0, 1\}$ for $\forall t$

☹ If $x(t_0) = \underset{1}{0} \Rightarrow x(t) = \underset{1}{0}$ is the unique sol with initial cond. $x(t_0) = 0$

$\Rightarrow x(0) = \underset{1}{0}$ a contradiction

— — —

$$\frac{1}{x(1-x)}\frac{dx}{dt} = 1$$

$$\int_0^T 1\, dt = \int_{0,\ x(0)=a}^{T,\ x(T)} \left(\frac{1}{x} + \frac{1}{1-x}\right)\frac{dx}{\cancel{dt}} \cdot \cancel{dt}$$

$$\Rightarrow t = \left[\ln\left|\frac{x}{1-x}\right|\right]_a^{x(t)}$$

$$t = \ln \left| \underbrace{\frac{x(t)}{1-x(t)}}_{1} \right| \left| \underbrace{\frac{1-a}{a}}_{2} \right|$$

$$\frac{x(0)}{1-x(0)} = \frac{a}{1-a}$$

1 & 2 have same sign

$$= \ln \frac{x}{1-x}\left(\frac{1}{a}-1\right)$$

$$e^{t} = \frac{x}{1-x}\left(\frac{1}{a}-1\right)$$

$$\left(\frac{1}{a}-1\right)e^{-t} = \frac{1-x}{x}$$

$$= \frac{1}{x}-1$$

$$\frac{1}{x} = 1+\left(\frac{1}{a}-1\right)e^{-t}$$

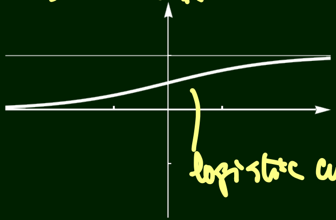
The logistic equation

discretization $\left(x_{n+1} - x_n = x_n(1-x_n) \right)$ a

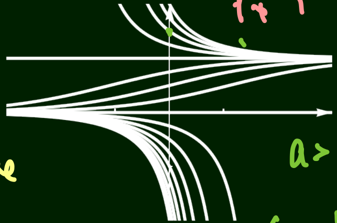
$x' = x(1-x), \qquad x(0) = a \qquad$ The Solution: $\quad x(t) = \dfrac{1}{1 + \left(\frac{1}{a} - 1\right) e^{-t}}$

<0

$a \in (0,1) \qquad t \in \mathbb{R}$



logistic curve



$1+$

$\vee$
0

$a > 1$

$0 \in \left\{ \log\left(1 - \frac{1}{a}\right) < t < \infty \right\}$

Problem 1-2

Problem (Ex. 1-2)

Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

the Frenet frame

$$e = \gamma'(s) \qquad \mathcal{F} = (e, n, b) : \mathbb{R} \to SO(3)$$

$$\mathcal{F}' = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

$$\mathcal{F}' = \frac{1}{1+s^2}\,\mathcal{F}\Omega_0 \qquad \Omega_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$s = \tan u \qquad \frac{ds}{du} = 1 + \tan^2 u = 1 + s^2$$

$$\begin{cases} \dfrac{d\mathcal{F}}{du} = \mathcal{F}\Omega_0 \\ \mathcal{F}(0) = \mathrm{id} \end{cases} \qquad \mathcal{F}(u) = \exp u\Omega_0 = \sum_{k=0}^{\infty} \frac{u^k}{k!}\Omega_0^k \quad \left(\begin{array}{l} 0\text{th term} \\ = \mathrm{id} \end{array}\right)$$

$$\Omega_0^2 = -\overbrace{\begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}}^{\Omega_1} \qquad \Omega_0^3 = -\Omega_0$$

$$\Omega_0^4 = -\Omega_1 \quad \cdots$$

$$J = \begin{pmatrix} \frac{1}{2}(1+\cos u) & -\frac{1}{\sqrt{2}}\sin u & \frac{1}{2}(1+\cos u) \\ \frac{1}{\sqrt{2}}\sin u & \cos u & -\frac{1}{\sqrt{2}}\sin u \\ \frac{1}{2}(1-\cos u) & -- & \end{pmatrix}$$

$\tan u = s$

$\cos u = \frac{1}{\sqrt{1+s^2}}$

$\sin u = \frac{s}{\sqrt{1+s^2}}$

$$\gamma' = \left(\frac{1}{2}(1+\cos u) \quad \frac{1}{\sqrt{2}}\sin u \quad \frac{1}{2}(1-\cos u) \right)$$

$$\gamma(s) = \left(\frac{1}{2}\left(s + \ln\left(s+\sqrt{1+s^2}\right)\right),\ \frac{1}{\sqrt{2}}\sqrt{1+s^2},\ \frac{1}{2}\left(s - \ln\left(s+\sqrt{1+s^2}\right)\right) \right)$$

Rmk $\frac{d}{ds}\Big(\underbrace{\frac{1}{\sqrt{2}}(\mathbb{e}+\mathbb{b})}_{\text{const.}}\Big) = \frac{1}{\sqrt{2}}\left(\kappa \mathbb{n} - \frac{\tau}{\kappa}\mathbb{n}\right) = 0$

If κ and τ : proportional $\left(\frac{\tau}{\kappa} = \text{const}\right)$

(a curve of constant slope)

$\Rightarrow \underbrace{a\mathbb{e} + b\,\mathbb{b}}_{\text{Darboux vector}}$: const.

(1st integral)