Advanced Topics in Geometry F1 (MTH.B506)

Integrablity Conditions

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Integrability Conditions

$P_0 \in U \subset \mathbb{R}^m$: a domain

$$\boxed{\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0.} \qquad (*)$$

Proposition (Prop. 2.2)

If a matrix-valued C^∞ function $X\colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies (*), it holds that

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j$$

for each (j, k) with $1 \leqq j < k \leqq m$.

· integrability conditions

$U \subset \mathbb{R}^2$: a domain $\qquad p \in U$

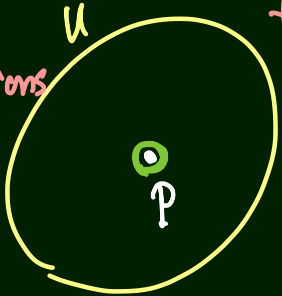
X : a matrix-valued unknown on U (square)

$m \times m$ unknown functions

$2 \times m^2$ relations

$$\begin{cases} \dfrac{\partial X}{\partial u} = X\Omega \\[2ex] \dfrac{\partial X}{\partial v} = X\Lambda \end{cases}$$

$X(p) = X_0$



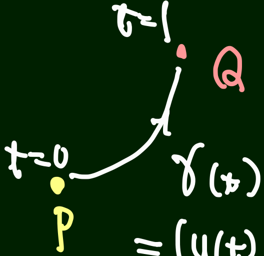
overdetermined system 過剰決定系

(Ω, Λ : given matrix-valued function)

$$\frac{\partial X}{\partial u} = X\Omega, \qquad \frac{\partial X}{\partial v} = X\Lambda, \qquad X(P) = X_0 \in \underbrace{GL(m, \mathbb{R})}_{\left(\substack{m\times m \\ \text{regular} \\ \text{matrices}}\right)}$$

$$\Rightarrow \quad X \in GL(m, \mathbb{R}) \text{ on } U$$

$$\hat{X} := X \circ \gamma$$



$$= (u(t), v(t))$$

$$\frac{d\hat{X}}{dt} = \frac{d}{dt} X(u(t), v(t))$$

$$= \dot{u}\frac{\partial X}{\partial u} + \dot{v}\frac{\partial X}{\partial v}$$

$$= \hat{X}\left(\dot{u}\hat{\Omega} + \dot{v}\hat{\Lambda}\right) \qquad \hat{\Omega} = \Omega \circ \gamma \text{ etc.}$$

$$\hat{X}(0) = X_0 \qquad \hat{X}(1) = X(Q)$$

$$\frac{d\hat{X}}{dt} = \hat{X}\,\Phi$$

$$\det \hat{X} = (\det X_0)\exp\int_0^t \mathrm{tr}\,\Phi\, d\tau$$

$$\neq 0 \qquad\qquad \neq 0$$

$$\therefore \det X(Q) = \det \hat{V}(1) \neq 0 \quad \lrcorner$$

✓ $X_0 \in SL(m,\mathbb{R})$, $\mathrm{tr}\,\Omega = \mathrm{tr}\,\Lambda = 0$

$$\Rightarrow X \in SL(m,\mathbb{R})$$

✓ $X_0 \in SO(m)$, ${}^t\Omega + \Omega = {}^t\Lambda + \Lambda = 0$

$$\Rightarrow X \in SO(m)$$

· $X: U \to GL(m, \mathbb{R})$ C^∞

· $\dfrac{\partial X}{\partial u} = X\Omega$, $\dfrac{\partial X}{\partial v} = X\Lambda$ $(*)$

· compatibility
· integrability
cond. of $(*)$

適合条件

可積分 regular

$$\Rightarrow \boxed{\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega}$$

$$\because \frac{\partial^2 X}{\partial v \partial u} = \frac{\partial}{\partial v}(X\Omega) = \frac{\partial X}{\partial v}\Omega + X\frac{\partial \Omega}{\partial v}$$

$$\| \qquad = X\Lambda\Omega + X\frac{\partial \Omega}{\partial v} = X\left(\Lambda\Omega + \frac{\partial \Omega}{\partial v}\right)$$

$$\frac{\partial^2 X}{\partial u \partial v} = \frac{\partial}{\partial u}(X\Lambda) = X\left(\Omega\Lambda + \frac{\partial \Lambda}{\partial u}\right)$$

(commutativity of partial derivatives.

Integrability Conditions = necessary cond. for $\exists X$
(sufficient ?)

Lemma (Lem. 2.3)

Let $\Omega_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

Then for each smooth map

$$\sigma \colon D \ni (t, w) \longmapsto \underline{\sigma(t, w)} = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

where $T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}$, $W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w}$, $(\widetilde{\Omega}_j := \Omega_j \circ \sigma)$.

Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

Lemma (Lem. 2.3)

Let $X\colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying (*). Then for each smooth path $\gamma\colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t)\frac{du^j}{dt}(t)\right)$$

on I, where $\gamma(t) = \big(u^1(t), \dots, u^m(t)\big)$.

Integrability of Linear systems

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{1}$$

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j\Omega_k - \Omega_k\Omega_j \tag{2}$$

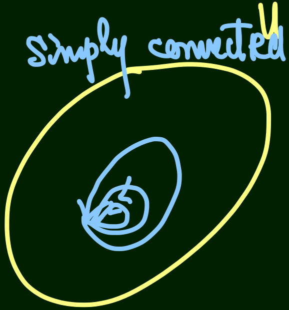
Theorem (Thm. 2.5)

Let $\Omega_j : U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ be C^∞-functions defined on a simply connected domain $U \subset \mathbb{R}^m$ satisfying (2). Then for each $\mathrm{P}_0 \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $X : U \to \mathrm{M}_n(\mathbb{R})$ satisfying (1)

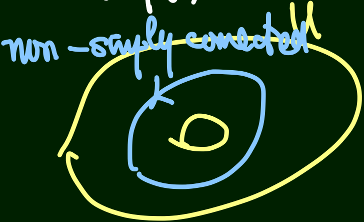
· integrability cond $\Rightarrow$ $\exists$ solution

(if U: simply connected) 単連結

- $\mathbb{R}^2$: simply connected
- $\mathbb{R}^2 \setminus \{(0,0)\}$ is not.
- $\mathbb{R}^3 \setminus \{(0,0,0)\}$ simp. conn



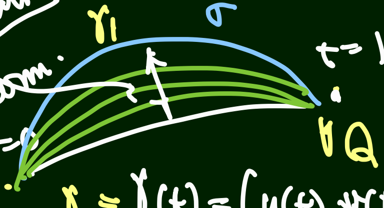
(each loop is homotopic to a pt.)



$\frac{\partial X}{\partial u} = X\Omega \qquad \frac{\partial X}{\partial v} = X\Lambda \qquad X(P) = X_0$

$\exists$ deformation

simple conn.



$\gamma_0 = \gamma(t) = (u(t), v(t))$

$\cdot\ \gamma_s$: a variation of paths

$\left(\hat{X} = \hat{X}(s,t)\right)$

Solve $\quad \frac{d\hat{X}}{dt} = \hat{X}\left(\dot{u}\,\Omega\circ\gamma + \dot{v}\,\Lambda\circ\gamma\right)$

$\hat{X}(0) = X_0$

$\boxed{\frac{\partial \hat{X}}{\partial s}(s,1) = 0}$

integrability

$\boxed{\hat{X}(1) \text{ does not depend a choice of } \gamma}$

$\Rightarrow$ Set $X(Q) := \hat{X}(1)$: a desired solution.

Application: Poincaré's lemma

Theorem (Poincaré's lemma)

If a differential 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* C^∞*-function* f *on* U *such that* $df = \omega$*. Such a function* f *is unique up to additive constants.*

$$\omega = \alpha\, du + \beta\, dv$$

$$df = \omega \quad : \quad (\alpha = f_u, \quad \beta = f_v)$$

$$d\omega = (\beta_u - \alpha_v)\, du \wedge dv$$

$$d\omega = 0 \iff \beta_u - \alpha_v = 0$$

integrability

$$\Omega$$

$$\begin{cases} \varphi_u = \alpha\, \varphi \\ \varphi_v = \hat{\beta}\, \varphi \end{cases} \qquad 1\times 1 \qquad \varphi(p) = 1$$

$$\exists\, \varphi > 0$$

$$f = \log \varphi$$

Application: Conjugation of harmonic functions

共役

Theorem

Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U. Then there exists a C^∞ harmonic function η on U such that $\underline{\xi(u,v) + \mathrm{i}\,\eta(u,v)}$ is holomorphic on U.

正則:

$\xi(u,v)$: harmonic (調和) $\qquad \xi_{uu} + \xi_{vv} = 0$

Cauchy-Riemann.

Solve $\begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$

$\omega = -\xi_v\,du + \xi_u\,dv$

$d\omega = 0$

$\Leftrightarrow \xi_{uu} + \xi_{vv} = 0$

Application: Conjugation of harmonic functions

Example

$\xi(u,v) = e^u \cos v$

conjugate harmonic fct

$$\left(\eta(u,v) = e^u \sin v\right)$$

$$\xi + i\eta = e^u(\cos v + i \sin v)$$
$$= e^{u+iv}$$

Exercise 2-1

Problem

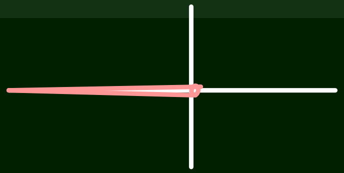
Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

1. *Show that* ξ *is harmonic on* U.
2. *Find the conjugate harmonic function* η *of* ξ *on*

$$V = \mathbb{R}^2 \setminus \{\underline{(u,0) \mid u \leqq 0}\} \subset U.$$ simply conn.

3. *Show that there exists no conjugate harmonic function of* ξ *defined on* U.

$\mathbb{R}^2$



Exercise 2-2

Problem

Consider a linear system of partial differential equations for 2×2-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$

$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

where (u, v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i, j = 1, 2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.