

# Advanced Topics in Geometry F1 (MTH.B506)

Integrability Conditions

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# Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \quad X(P_0) = X_0. \quad (*)$$

## Proposition (Prop. 2.2)

If a matrix-valued  $C^\infty$  function  $X: U \rightarrow \text{GL}(n, \mathbb{R})$  satisfies  $(*)$ , it holds that

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j$$

for each  $(j, k)$  with  $1 \leq j < k \leq m$ .

# Integrability Conditions

## Lemma (Lem. 2.3)

Let  $\Omega_j: U \rightarrow M_n(\mathbb{R})$  ( $j = 1, \dots, m$ ) be  $C^\infty$ -maps defined on a domain  $U \subset \mathbb{R}^m$  which satisfy

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

Then for each smooth map

$$\sigma: D \ni (t, w) \mapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain  $D \subset \mathbb{R}^2$ , it holds that

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

where  $T := \sum_{j=1}^m \tilde{\Omega}_j \frac{\partial u^j}{\partial t}$ ,  $W := \sum_{j=1}^m \tilde{\Omega}_j \frac{\partial u^j}{\partial w}$ , ( $\tilde{\Omega}_j := \Omega_j \circ \sigma$ ).

# Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \quad X(P_0) = X_0. \quad (*)$$

## Lemma (Lem. 2.3)

Let  $X: U \rightarrow M_n(\mathbb{R})$  be a  $C^\infty$ -map satisfying (\*). Then for each smooth path  $\gamma: I \rightarrow U$  defined on an interval  $I \subset \mathbb{R}$ ,  $\hat{X} := X \circ \gamma: I \rightarrow M_n(\mathbb{R})$  satisfies the ordinary differential equation

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left( \Omega_\gamma(t) := \sum_{j=1}^m \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t) \right)$$

on  $I$ , where  $\gamma(t) = (u^1(t), \dots, u^m(t))$ .

# Integrability of Linear systems

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \quad X(P_0) = X_0. \quad (1)$$

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j \quad (2)$$

## Theorem (Thm. 2.5)

Let  $\Omega_j: U \rightarrow M_n(\mathbb{R})$  ( $j = 1, \dots, m$ ) be  $C^\infty$ -functions defined on a simply connected domain  $U \subset \mathbb{R}^m$  satisfying (2). Then for each  $P_0 \in U$  and  $X_0 \in M_n(\mathbb{R})$ , there exists the unique  $n \times n$ -matrix valued function  $X: U \rightarrow M_n(\mathbb{R})$  satisfying (1)

# Application: Poincaré's lemma

## Theorem (Poincaré's lemma)

*If a differential 1-form*

$$\omega = \sum_{j=1}^m \alpha_j(u^1, \dots, u^m) du^j$$

*defined on a simply connected domain  $U \subset \mathbb{R}^m$  is closed, that is,  $d\omega = 0$  holds, then there exists a  $C^\infty$ -function  $f$  on  $U$  such that  $df = \omega$ . Such a function  $f$  is unique up to additive constants.*

## Application: Conjugation of harmonic functions

### Theorem

*Let  $U \subset \mathbb{C} = \mathbb{R}^2$  be a simply connected domain and  $\xi(u, v)$  a  $C^\infty$ -function harmonic on  $U$ . Then there exists a  $C^\infty$  harmonic function  $\eta$  on  $U$  such that  $\xi(u, v) + i\eta(u, v)$  is holomorphic on  $U$ .*

# Application: Conjugation of harmonic functions

## Example

$$\xi(u, v) = e^u \cos v$$



## Exercise 2-1

### Problem

Let  $\xi(u, v) := \log \sqrt{u^2 + v^2}$  be a function defined on  $U := \mathbb{R}^2 \setminus \{(0, 0)\}$ .

- 1 Show that  $\xi$  is harmonic on  $U$ .
- 2 Find the conjugate harmonic function  $\eta$  of  $\xi$  on

$$V = \mathbb{R}^2 \setminus \{(u, 0) \mid u \leq 0\} \subset U.$$

- 3 Show that there exists no conjugate harmonic function of  $\xi$  defined on  $U$ .

## Exercise 2-2

### Problem

Consider a linear system of partial differential equations for  $2 \times 2$ -matrix valued unknown  $X$  on a domain  $U \subset \mathbb{R}^2$  as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$
$$\left( \Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right),$$

where  $(u, v)$  are the canonical coordinate system of  $\mathbb{R}^2$ , and  $\alpha, \beta$  and  $h_j^i$  ( $i, j = 1, 2$ ) are smooth functions defined on  $U$ . Write down the integrability conditions in terms of  $\alpha, \beta$  and  $h_j^i$ .