Advanced Topics in Geometry F1 (MTH.B506)

Differential Forms

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-f1/

Tokyo Institute of Technology

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Problem 2-1

$\xi + i\eta = \log z$ $\quad z = re^{i\theta}$

$= \log r + i\theta$

Problem

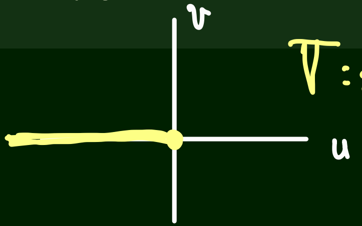
Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ *be a function defined on* $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

non simply connected

1. *Show that* ξ *is harmonic on* U.
2. *Find the conjugate harmonic function* η *of* ξ *on*

$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of* ξ *defined on* U.



V : simply conn

$$\xi = \lg\sqrt{u^2+v^2} \qquad \xi_u = \frac{u}{u^2+v^2}$$

$$\xi_{uu} = \frac{1}{u^2+v^2} - \frac{2u^2}{(u^2+v^2)^2}$$

$$+$$

$$0 = \eta_{vv} = \frac{1}{u^2+v^2} - \frac{2v^2}{(u^2+v^2)^2}.$$

$$\begin{cases} \eta_u = -\xi_v \\ \eta_v = \xi_u \end{cases}$$

$(u>0)$

$$\eta_u = \frac{-v}{u^2+v^2} = \frac{-\frac{v}{u^2}}{1+\frac{v^2}{u^2}}$$

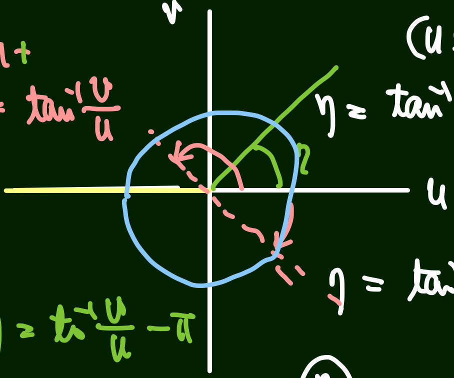
$$= \frac{1}{1+\frac{v^2}{u^2}}\left(\frac{v}{u}\right)_u$$

$$\eta = \tan^{-1}\frac{v}{u}\ (+\,\mathrm{const}) \qquad = \left(\tan^{-1}\frac{v}{u}\right)_u$$

$$\eta_v = \frac{u}{u^2+v^2} = \frac{\frac{1}{u}}{1+\frac{v^2}{u^2}} = \left(\tan^{-1}\frac{v}{u}\right)_v$$

$\pi +$

$\eta = \tan^{-1}\frac{v}{u}$



$(u>0)$

$\eta = \tan^{-1}\frac{v}{u} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\eta = \tan^{-1}\frac{v}{u}$

$\left(\operatorname{atan2}(u, v)\right)$

$\eta = \tan^{-1}\frac{v}{u} - \pi$

η

2π-periodic

* η cannot be extended to $\mathbb{C}\setminus\{0\}$

Assume $\exists\, \eta$ on $\mathbb{C}\setminus\{0\}$: Set $\hat{\eta}(\theta) = \eta(\cos\theta, \sin\theta)$

$$\frac{d}{d\theta}\hat{\eta} = -\sin\theta\, \eta_u + \cos\theta\, \eta_v = 1$$

$\hat{\eta}(\theta) = \theta + \text{const}$

not periodic

Problem 2-2

Problem

Consider a linear system of partial differential equations for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$
$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

where (u, v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i, j = 1, 2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \quad \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

(Rem Ω, Λ : skew-symmetric
$\Omega^T = -\Omega, \quad \Lambda^T = -\Lambda$)

Integrability : skew

$$\Omega_v - \Lambda_u - \Omega\Lambda + \Lambda\Omega = 0$$: 3-equalities ⇕

* $\Omega_v - \Lambda_u$: skew-symm
* $\Omega\Lambda - \Lambda\Omega$: skew-symm

$$\alpha_v - \beta_u = h_1^1 h_2^2 - h_2^1 h_1^2$$
$$h^1_{1v} - h^1_{2u} = \beta h^2_1 - \alpha h^2_2$$
$$h^2_{1v} - h^2_{2u} = -\beta h^1_1 + \alpha h^1_2$$

Gauss

Codazzi

in the fundamental theorem for surfaces

coefficients of the connection

- $X_u = X\Omega$, $X_v = X\Lambda$ (depends on choice of coordinates (u,v) on $\mathbb{R}^2$)

"coordinate free form"

$$dX = X\hat{\Omega}$$

(LHS: $X_u\,du + X_v\,dv$: (differential) 1 form

RHS $\hat{\Omega} = \Omega\,du + \Lambda\,dv$)

curvature form

integrability: $\left(d\hat{\Omega} + \hat{\Omega}\wedge\hat{\Omega} = 0 \right)$

2-form