Advanced Topics in Geometry F1 (MTH.B506)

Differential Forms

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2023/06/27

Problem 2-1

Problem

Let $\xi(u, v) := \log \sqrt{u^2 + v^2}$ be a function defined on $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

1. *Show that ξ is harmonic on U.*

2. *Find the conjugate harmonic function η of ξ on*

$$V = \mathbb{R}^2 \setminus \{(u, 0) \,|\, u \leqq 0\} \subset U.$$

3. *Show that there exists no conjugate harmonic function of ξ defined on U.*

Problem 2-2

Problem

Consider a linear system of partial differential equations for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda,$$

$$\left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix} \right),$$

where (u, v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h_j^i $(i, j = 1, 2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h_j^i.