Advanced Topics in Geometry F1 (MTH.B506)

Differential Forms

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Notation

- M: an n-dimensional manifold.
- g: a Riemannian metric on M.
- TM: the tangent bundle of M. ~ $\bigcup_{p\in M} T_pM$
- T^*M: the cotangent bundle of M. — $\bigcup_{p\in M} T_p^*M$
- $\mathcal{F}(M)$: the set of C^∞-functions.
- $\mathfrak{X}(M)$: the set of C^∞ vector fields. : $\mathcal{F}(M)$-module
- $(x^1, \ldots, x^n)$: a local coordinate system around $p \in M$.
- $\left(\dfrac{\partial}{\partial x^j}\right)_p \in T_pM$. · $\left\{\left(\dfrac{\partial}{\partial x^i}\right)_p\right\}$: a basis
- $(dx^j)_p \in T_p^*M$ · $\{dx^i_p\}$: a basis

$T_p^*M = \{\alpha : T_pM \to \mathbb{R}\}$ (linear), n-dim

$f\,X$ (func) (vect field)

$Xf \in \mathcal{F}(M)$ (derivative)

$$dx^j\left(\frac{\partial}{\partial x^k}\right) = \delta^j_k \quad \text{(Kronecker's delta)}$$

Lie Brackets (Review)

" $[X,Y] = XY - YX$ "

$X, Y \in \mathfrak{X}(M)$ $\qquad [X,Y] \in \mathfrak{X}(M)$

$$[X,Y]f = X(Yf) - Y(Xf)$$

$[X,Y] = -[Y,X]$

Lemma $X, Y \in \mathfrak{X}(M)$, $f \in \mathfrak{F}(M)$

$$[fX,Y] = f[X,Y] - (Yf)X, \qquad [X,fY] = f[X,Y] + (Xf)Y.$$

$$\begin{aligned}
[fX,Y]\varphi &= fX(Y\varphi) - Y(fX\varphi) \\
&= f(X(Y\varphi)) - Y(f \cdot X\varphi) \\
&= f(X(Y\varphi)) - (Yf)\cdot X\varphi - fY(X\varphi) \\
&= f[X,Y]\varphi - \{(Yf)\cdot X\}\varphi
\end{aligned}$$

the set of sections

linear

$\{T_pM \to \mathbb{R}\}$

- T^*M
- $\wedge^1(M) := \Gamma(T^*M) \ni \alpha$

$\alpha : M \to T^*M$

$p \mapsto \alpha_p \in T_p^*M$

1-form

Lemma $\mathbb{R}$

A linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ is a 1-form if and only if

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M))$$

holds.

$\mathcal{F}(M)$-linear

inverse

$\star$ $\alpha \in \Lambda^1(M) \Rightarrow \alpha : \mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$

$\alpha(X)(p) = \alpha_p(X_p)$ $\mathcal{F}(M)$-linear

$\alpha(fX) = f\alpha(X)$

Rem $X \in \mathfrak{X}(M)$: fix

$$\alpha : \mathfrak{X}(M) \ni Y \xmapsto[\text{linear}]{} \alpha(Y) = \langle [X, Y], X \rangle \in \mathfrak{F}(M)$$

$\Lambda^1(M)$

$$\begin{aligned}
\underline{\alpha(fY)} &= \langle [X, fY], X \rangle \\
&= \langle f[X, Y] + (Xf)Y, X \rangle \\
&= f\langle [X, Y], X \rangle + (Xf)\langle Y, X \rangle \\
&= \underline{f\alpha(Y)} + (Xf)\langle Y, X \rangle
\end{aligned}$$

not $\mathfrak{F}(M)$-linear.

Tensors

tensor product

$T_pM \times T_pM \longrightarrow \mathbb{R}$

bilinear (双线性)

- $T^*M \otimes T^*M := \bigcup_{p\in M} \boxed{T_p^*M \otimes T_p^*M}$
- $\Gamma(T^*M \otimes T^*M)$ covariant 2-tensors. 共变
- $\beta \in \wedge^2(M)$ skew-symm

$\cap$

$\Gamma(T^*M \otimes T^*M)$

2-forms

$\mathfrak{F}(M)$-bilinear

$$\begin{aligned} \beta : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{F}(M) \\ (X, Y) &\longmapsto \beta(X, Y) \end{aligned}$$

skew-symm $\quad \beta(Y, X) = -\beta(X, Y)$

Tensors

- $T^*M \otimes T^*M \otimes T^*M := \bigcup_{p \in M} T_p^*M \otimes T_p^*M \otimes T_p^*M$
- $\Gamma(T^*M \otimes T^*M \otimes T_p^*M)$
- $\wedge^3(M)$

3-forms

skew-symm

$\omega(X, Y, Z) = -\omega(Y, X, Z)$ --- etc.

Exterior products. (外積)

- α, β, $\gamma \in \wedge^1(M)$
- $\omega \in \wedge^2(M)$

$\alpha \wedge \beta = -\beta \wedge \alpha$

$\alpha \wedge \beta \in \Lambda^2(M)$

$$(\alpha \wedge \beta)(X, Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X).$$

$$\begin{aligned}(\alpha \wedge \omega)(X, Y, Z) &= (\omega \wedge \alpha)(X, Y, Z) \\ &:= \alpha(X, Y)\omega(Z) + \alpha(Y, Z)\omega(X) + \alpha(Z, X)\omega(Y).\end{aligned}$$

$\alpha \wedge \omega \in \Lambda^3(M)$

Lemma

$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma$

Exterior derivative. (外微分)

- $f \in \wedge^0(M) = \mathcal{F}(M)$, $\alpha, \cancel{\beta} \in \wedge^1(M)$, $\cancel{\alpha}\ \beta \in \wedge^2(M)$.

$$df : \mathfrak{X}(M) \ni X \mapsto df(X) = Xf \in \mathcal{F}(M), \qquad df \in \wedge^1(M) \ *$$

$$d\alpha : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \in \mathcal{F}(M) \qquad d\alpha \in \wedge^2(M)$$

$$d\beta : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y, Z) \mapsto \qquad d\beta \in \wedge^3(M)$$
$$X\beta(Y, Z) + Y\beta(Z, X) + Z\beta(X, Y) - \beta([X, Y], Z) - \beta([Y, Z], X) - \beta([Z, X], Y) \in \mathcal{F}(M).$$

Lemma $\alpha, \beta \in \wedge^1(M)$

$dd\alpha = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$,

$* \ df(\varphi X) = \varphi X f = \varphi\, df(X)$

$df(X) = Xf$

$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$

- When $\alpha = df$

$$X\alpha(Y) - Y\alpha(X) = X(Yf) - Y(Xf)$$

$$= [X, Y]f$$

$$= df([X, Y])$$

$d\alpha = 0$

Riemannian connection

(M, g): a Riemannian mfd.

Lemma

$\exists 1 \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ *with*

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle$$

torsion free

parallelity of g (平行性)

for $X, Y, Z \in \mathfrak{X}(M)$.)

- ∇: the Riemannian connection, the Levi-Civita connection

Lemma

∇ is a linear connection

- $\nabla_{fX} Y = f \nabla_X Y$, $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$.

in particular, $X \longmapsto \nabla_X Y$ is a 1-form

fix

$e_j \in \mathfrak{X}(U)$

- $U \subset M$: a domain　正規直交枠
- $\{e_1, \ldots, e_n\}$: an orthonormal frame.　$\langle e_j, e_k \rangle = \delta_{jk}$

$$\omega_j(X) := \langle e_j, X \rangle \qquad \text{(the dual frame)}$$

ω^j　　j

$$\omega^j : \mathfrak{X}(U) \longrightarrow \mathfrak{F}(U) \qquad \omega^j \in \Lambda^1(U)$$

$$\omega^j(e_k) = \delta^j_k$$

Gauge transformations

- $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ and $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$: two orthonormal frames on U.
- $\Theta\colon U \to \mathrm{O}(n)$: the Gauge transformation: ゲージ変換

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta. \quad \in \mathrm{O}(n)$$

- $(\omega^1, \ldots, \omega^n)$, $(\lambda_1, \ldots, \lambda_n)$: the duals.

$$\begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \Theta \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}.$$

$$\mathrm{id} = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} (\boldsymbol{e}_1 \cdots \boldsymbol{e}_n) = \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} (\boldsymbol{v}_1 \sim \boldsymbol{v}_n)$$

Connection Forms

- (M, g): a Riemannian manifold, ∇: the Levi-Civita connection.
- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$: an orthornomal frame on $U \subset M$.
- $(\omega^1, \dots, \omega^n)$: its dual.

the connection form(s)

$$\Omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \dots & \omega_n^1 \\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{pmatrix}, \qquad \omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \wedge^1(U).$$

vetor-valued

$$\nabla e_j = \sum_k \underbrace{\omega_j^k}_{\text{connection form}} e_k$$

1-form

$$\nabla [e_1 \cdots e_n] = [e_1 \cdots e_n]\Omega$$

Connection Forms

$\omega_j^k := \langle \nabla e_j, e_k \rangle \in \wedge^1(U)$.

Lemma

- $\omega_j^k = -\omega_k^j$.
- $d\omega^i = \sum_{l=1}^n \omega^l \wedge \omega_l^i$.

show symm $\Leftarrow$ $X\langle Y, Z\rangle$

$= \langle \nabla_X Y, Z\rangle$

$+ \langle Y, \nabla_X Z\rangle$

(Next week)

$\nabla_X Y - \nabla_Y X = [X, Y]$

Exercise 3-1

Problem

Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta.$$

Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$.

Formula of the Gauge transf.

Exercise 3-2

Problem

Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-1)$ the hyperbolic 2-space (i.e. the hyperbolic plane) of constant curvature -1.

1. *Verify that gives a local coordinate system on $U := H^2(-1) \setminus \{(1,0,0)\}$, and*

$$\begin{cases} \boldsymbol{e}_1 := (\sinh u, \cos v \cosh u, \sin v \cosh u), \\ \boldsymbol{e}_2 := (0, -\sin v, \cos v) \end{cases}$$

   forms a orthonormal frame on U.

2. *Compute the connection form(s) with respect to the orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.*

☆ ◯ du + ◯ dv