Advanced Topics in Geometry F1 (MTH.B506)

Differential Forms

Kotaro Yamada
kotaro@math.titech.ac.jp
http://www.math.titech.ac.jp/~kotaro/class/2023/geom-f1/

Tokyo Institute of Technology

2023/06/27 (2023/04/25 訂正)

Notation

- M: an n-dimensional manifold.
- g: a Riemannian metric on M.
- TM: the tangent bundle of M.
- T^*M: the cotangent bundle of M.
- $\mathcal{F}(M)$: the set of C^∞-functions.
- $\mathfrak{X}(M)$: the set of C^∞ vector fields.
- $(x^1, \ldots, x^n)$: a local coordinate system around $p \in M$.
- $\left(\frac{\partial}{\partial x^j}\right)_p \in T_pM$.
- $\left(dx^j\right)_p \in T_p^*M$

Lie Brackets (Review)

$$[X, Y]f = X(Yf) - Y(Xf)$$

Lemma

$$[fX, Y] = f[X, Y] - (Yf)X, \qquad [X, fY] = f[X, Y] + (Xf)Y.$$

Tensors

- T^*M
- $\wedge^1(M) := \Gamma(T^*M)$

Lemma

A linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ *is a* 1*-form if and only if*

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M))$$

holds.

Tensors

- $T^*M \otimes T^*M := \bigcup_{p \in M} T_p^*M \otimes T_p^*M$
- $\Gamma(T^*M \otimes T^*M)$
- $\wedge^2(M)$

Tensors

- $T^*M \otimes T^*M \otimes T^*M := \bigcup_{p \in M} T_p^*M \otimes T_p^*M \otimes T_p^*M$
- $\Gamma(T^*M \otimes T^*M \otimes T_p^*M)$
- $\wedge^3(M)$

Exterior products.

- $\alpha,\ \beta,\ \gamma \in \wedge^1(M)$
- $\omega \in \wedge^2(M)$

$$
\begin{aligned}
(\alpha \wedge \beta)(X, Y) &:= \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \\
(\alpha \wedge \omega)(X, Y, Z) &= (\omega \wedge \alpha)(X, Y, Z) \\
&:= \alpha(X, Y)\omega(Z) + \alpha(Y, Z)\omega(X) + \alpha(Z, X)\omega(Y).
\end{aligned}
$$

Lemma

$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

Exterior derivative.

- $f \in \wedge^0(M) = \mathcal{F}(M)$, α, $\beta \in \wedge^1(M)$, $\omega \in \wedge^2(M)$.

$$
\begin{aligned}
df &\colon \mathfrak{X}(M) \ni X \mapsto df(X) = Xf \in \mathcal{F}(M), \\
d\alpha &\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \\
&\quad X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \in \mathcal{F}(M) \\
d\beta &\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y, Z) \mapsto \\
&\quad X\beta(Y, Z) + Y\beta(Z, X) + Z\beta(X, Y) \\
&\quad - \beta([X, Y], Z) - \beta([Y, Z], X) - \beta([Z, X], Y) \in \mathcal{F}(M)
\end{aligned}
$$

Lemma

$dd\alpha = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$,

Riemannian connection

Lemma

$\exists 1 \nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ *with*

$$\nabla_X Y - \nabla_Y X = [X,Y], \qquad X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle$$

for $X, Y, Z \in \mathfrak{X}(M)$*.)*

- ∇: the Riemannian connection, the Levi-Civita connection

Lemma

$\nabla_{fX} Y = f\nabla_X Y$, $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$.

Orthonormal frame

- $U \subset M$: a domain
- $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$: an orthonormal frame.

$$\omega_j(X) := \langle \boldsymbol{e}_1, X \rangle \qquad \text{(the dual frame)}$$

Gauge transformations

- $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ and $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$: two orthonormal frames on U.
- $\Theta\colon U \to \mathrm{O}(n)$: the Gauge transformation:

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta.$$

- $(\omega^1, \ldots, \omega^n)$, $(\lambda_1, \ldots, \lambda_n)$: the duals.

$$\begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \Theta \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}.$$

Connection Forms

- (M, g): a Riemannian manifold, ∇: the Levi-Civita connection.
- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \dots, \omega^n)$: its dual.

$$\Omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \dots & \omega_n^1 \\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{pmatrix}, \qquad \omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \wedge^1(U).$$

Connection Forms

$\omega_j^k := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle \in \wedge^1(U)$.

Lemma

- $\omega_j^k = -\omega_k^j$.
- $d\omega^i = \sum_{l=1}^n \omega^l \wedge \omega_l^i$.

Exercise 3-1

Problem

Let $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on a domain* U *of a Riemannian* n*-manifold* M*, which are related as*

$$[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]\Theta.$$

Show that the connection forms Ω *of* $\{\boldsymbol{e}_j\}$ *and* Λ *of* $\{\boldsymbol{v}_j\}$ *satisfy* $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$*.*

Exercise 3-2

Problem

Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-1)$ the hyperbolic 2-space (i.e. the hyperbolic plane) of constant curvature -1.

1. *Verify that gives a local coordinate system on $U := H^2(-1) \setminus \{(1,0,0)\}$, and*

$$\boldsymbol{e}_1 := (\sinh u, \cos v \cosh u, \sin v \cosh u),$$
$$\boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

   forms a orthonormal frame on U.

2. *Compute the connection form(s) with respect to the orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.*