Advanced Topics in Geometry F1 (MTH.B506)

Curvature form

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2023/07/04 (2023/04/25 訂正)

Review

- (M, g): a Riemannian n-manifold; $\langle\ ,\ \rangle$: the inner product w. r. to g.
- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \dots, \omega^n)$: the dual frame of $(\boldsymbol{e}_j)$:

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

- ∇: the Levi-Civita connection on (M, g)
- $\Omega := (\omega_i^j)$: the connection form:

$$\nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega$$

- $\omega_j^k = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$

The Connection forms

Lemma (Lemmas 3.16/17)

- $\omega_j^k = -\omega_k^j$, *i. e.* $\Omega^T + \Omega = O$
- $d\omega^i = \sum_{l=1}^{n} \omega^l \wedge \omega_l^i$

cf.

- $X \langle Y, Z \rangle = \langle \nabla_X Y, \nabla Z \rangle + \langle Y, \nabla_X Z \rangle$
- $\nabla_X Y - \nabla_Y X = [X, Y]$

The Connection forms

Lemma (Proposition 4.2)

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(-\langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle + \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i \rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j \rangle \Big).$$

Gauge Transformasions

Lemma (Ex. 3-1; Prop. 4.10 (1))

- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ *and* $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$*: orthonormal frames*
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta$ $(\Theta \in \mathrm{SO}(n))$
- Ω*: the connection form w. r. to* $[\boldsymbol{e}_j]$
- $\widetilde{\Omega}$*: the connection form w. r. to* $[\boldsymbol{v}_j]$

$\Rightarrow$

$$\Omega = \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta.$$

The Curvature Forms

Definition

$K := d\Omega + \Omega \wedge \Omega$: curvature form w. r. to $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$.

Gauge Transformasions

Lemma (Prop. 4.10 (2))

- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$ *and* $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$*: orthonormal frames*
- $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta$ $(\Theta \in \mathrm{SO}(n))$
- Ω, Λ*: the connection forms of* $(\boldsymbol{e}_j)$ *and* $(\boldsymbol{v}_j)$*, resp.*
- K, $\widetilde{K}$*: the curvature forms of* $(\boldsymbol{e}_j)$ *and* $(\boldsymbol{v}_j)$*, resp.*

$\Rightarrow$

$$K = \Theta^{-1}\widetilde{K}\Theta$$

Flatness

(M, g) is flat $\Leftrightarrow K = 0$

Theorem (Thm. 4.11)

(M, g): flat $\Rightarrow$
$\forall p \in M$, $\exists(U; x^1, \ldots, x^n)$: a chart around p such that

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$$

is an orthonormal frame on U.

Exercise 4-1

Problem

Consider a Riemannian metric $g = dr^2 + \{\varphi(r)\}^2\, d\theta^2$ *on* $U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\}$, *where* $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \varphi'(r) = 1.$$

Find a function φ *such that* (U, g) *is flat.*
(Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ *is an orthonormal frame.)*

Exercise 4-2

Problem

Compute the curvature form of $H^2(-1)$ *with respect to an orthonormal frame* $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ *as in Exercise 3-2.*