Advanced Topics in Geometry F1 (MTH.B506)

Sectional Curvature

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Problem 4-1

$\mathbb{R}_+ \times S^1$

warped product metric.

Problem

Consider a Riemannian metric $\boxed{g = dr^2 + \{\varphi(r)\}^2 d\theta^2}$ *on* $U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\}$, *where* $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \varphi'(r) = 1.$$

Find a function φ *such that* (U, g) *is flat.*
Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ *is an orthonormal frame.*

$g = dr^2 + \varphi^2 d\theta^2 \qquad \varphi = \varphi(r)$

$e_1 = \frac{\partial}{\partial r} \qquad e_2 = \frac{1}{\varphi}\frac{\partial}{\partial \theta} \Rightarrow \{e_1, e_2\}$: orthonormal.

Lie bracket

$$[e_1, e_2]f = e_1(e_2 f) - e_2(e_1 f)$$

$$= \frac{\partial}{\partial r}\Big(\underbrace{\frac{1}{\varphi}}\underbrace{\frac{\partial}{\partial \theta} f}\Big) - \frac{1}{\varphi}\frac{\partial}{\partial \theta}\cdot\frac{\partial}{\partial r} f$$

$$= -\frac{\dot{\varphi}}{\varphi^2} f_\theta + \cancel{\frac{1}{\varphi} f_{\theta r} - \frac{1}{\varphi} f_{r\theta}}$$

$$= -\frac{\dot{\varphi}}{\varphi}\frac{1}{\varphi}\frac{\partial}{\partial \theta} f = -\frac{\dot{\varphi}}{\varphi} e_2 f$$

$$[e_1, e_2] = -\frac{\dot{\varphi}}{\varphi} e_2$$

Connection form

$\nabla_X Y - \nabla_Y X = [X, Y]$

$$\langle \nabla_{e_1} e_1, e_1 \rangle = \frac{1}{2} e_1 \overbrace{\langle e_1, e_1 \rangle}^{1} = 0$$

$$\langle \nabla_{e_1} e_1, e_2 \rangle = e_1 \overbrace{\langle e_1, e_2 \rangle}^{0} - \langle e_1, \nabla_{e_1} e_2 \rangle$$

$$= - \langle e_1, \nabla_{e_1} e_2 \rangle = - \cancel{\langle e_1, \nabla_{e_2} e_1 \rangle} \quad \tfrac{1}{2} e_2 \langle e_1, e_1 \rangle$$

$$- \langle e_1, [e_1, e_2] \rangle$$

$$= - \langle e_1, [e_1, e_2] \rangle = \langle e_1, \frac{\varphi'}{\varphi} e_2 \rangle = 0$$

$$\therefore \nabla_{e_1} e_1 = 0$$

$$\frac{\varphi'}{\varphi} e_2 \qquad \frac{\varphi'}{\varphi}$$

$$\langle \nabla_{e_2} e_1, e_1 \rangle = 0$$

$$\langle \nabla_{e_2} e_1, e_2 \rangle = \langle \cancel{\nabla_{e_1} e_2}, e_2 \rangle + \langle [e_2, e_1], e_2 \rangle$$

$$\nabla_{e_1} e_1 = 0 \qquad \nabla_{e_2} e_1 = \frac{\dot\varphi}{\varphi} e_2 \quad \underline{\text{dual}} \quad \begin{matrix} \omega^1 \\ \omega^2 \end{matrix}$$

$$\nabla_X e_1 = \underbrace{\langle X, e_2 \rangle}_{= \, \omega^2(X)} \frac{\dot\varphi}{\varphi} e_2$$

$$\nabla e_1 = \omega^2 \frac{\dot\varphi}{\varphi} e_2 = \omega_1^1 e_1 + \omega_1^2 e_2$$

$$\omega_1^2 = \frac{\dot\varphi}{\varphi} \omega^2$$

$$\omega_2^1 = -\omega_1^2 = -\frac{\dot\varphi}{\varphi} \omega^2$$

$$\Omega = \frac{\dot\varphi}{\varphi}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\omega^2$$

2×2 skew symmetric

$$K = d\Omega + \overset{=0}{\Omega\wedge\Omega}$$

$$\omega^2 = \langle \cdot \;\; e_2\rangle = \frac{1}{\varphi}\left\langle \cdot \;\; \frac{\partial}{\partial\theta}\right\rangle$$

$$= \frac{1}{\varphi}\,d\theta$$

$$K = d\left\{\begin{pmatrix}0 & -\dot\varphi\\ \dot\varphi & 0\end{pmatrix}d\theta\right\} = \begin{pmatrix}0 & -\ddot\varphi\\ \ddot\varphi & 0\end{pmatrix}\underline{dr\wedge d\theta}$$

Flat $\quad \ddot\varphi = 0 \qquad \varphi(r) = r$

$\uparrow$

bdry cond

$g = dr^2 + r^2 d\theta^2$

: Riem metric of the Euclidean plane under the polar coordinate system.

Problem 4-2

Problem

Compute the curvature form of $H^2(-1)$ with respect to an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ as in Exercise 3-2.

$$\begin{aligned}
\boldsymbol{f} &= (\cosh u, \cos v \sinh u, \sin v \sinh u) \colon (0,\infty) \times (-\pi,\pi) \to \mathbb{R}^3_1. \\
\boldsymbol{e}_1 &= (\sinh u, \cos v \cosh u, \sin v \cosh u), \\
\boldsymbol{e}_2 &= (0, -\sin v, \cos v)
\end{aligned}$$

$ds^2 = du^2 + \sinh^2 u \, dv^2$: warped product

$$\Rightarrow \quad \underline{\underline{K}} = -\frac{\varphi''}{\varphi}\,\omega^1 \wedge \omega^2 \underset{\substack{\uparrow \\ \varphi = \sinh u}}{=} -\underline{\underline{\omega^1 \wedge \omega^2}}$$