Advanced Topics in Geometry F1 (MTH.B506)

Sectional Curvature

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Review

- (M, g): a Riemannian n-manifold; $\langle\ ,\ \rangle$: the inner product w. r. to g.
- $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \ldots, \omega^n)$: the dual frame of $(\boldsymbol{e}_j)$:

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

- ∇: the Levi-Civita connection on (M, g)
- $\Omega := (\omega_i^j)$: the connection form:

$$\nabla[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\Omega$$

- $\omega_j^k(x) = \langle \nabla_x \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form
- $\kappa_i^j = d\omega_i^j + \sum_l \omega_l^j \wedge \omega_i^l$. 2-form.

Review

- $\omega_j^k = -\omega_k^j$, $\kappa_j^k = -\kappa_k^j$ $\leftarrow X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$
- $d\omega^i = \sum_{l=1}^{n} \omega^l \wedge \omega_l^i$ $\leftarrow \nabla_X Y - \nabla_Y X = [X, Y]$

cf.

- $X\langle Y, Z\rangle = \langle \nabla_X Y, \nabla Z\rangle + \langle Y, \nabla_X Z\rangle$
- $\nabla_X Y - \nabla_Y X = [X, Y]$

Proposition (The first Bianchi identity; Prop. 5.2)

$$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) + \kappa^i_k(\boldsymbol{e}_l, \boldsymbol{e}_j) + \kappa^i_l(\boldsymbol{e}_j, \boldsymbol{e}_k) = 0.$$

$\because\ dd\omega^i = 0.$ $(e_a \quad e_b \quad e_c)$

$$d\left(\sum \omega^l \wedge \omega^i_l\right) = \sum \left(d\omega^l \wedge \omega^i_l - \omega^l \wedge d\omega^i_l \right)$$

$$d\omega^l = \sum \omega^m \wedge \omega^l_m .$$

$$d\omega^i_l = \kappa^i_l - \sum \omega^m_l \wedge \omega^i_m$$

The Bianchi identity

Corollary (Cor. 5.3)

$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa^k_l(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

The bilinear form derived from the curvature form

indep of $\{e_i\}$

bilinear form on $T_pM \wedge T_pM$

$$K(\xi, \eta) := \sum_{i<j, k<l} \underline{\kappa_i^j(e_k, e_l)\xi^{kl}\eta^{ij}},$$

$\in \underline{T_pM \wedge T_pM}$

$$\xi = \sum_{k<l} \xi^{kl} \underbrace{e_k \wedge e_l}, \quad \eta = \sum_{i<j} \eta^{ij} e_i \wedge e_j$$

Lemma

K is symmetric.

Cor 5.3

$T_pM \wedge T_pM = \mathrm{Span}\{e_i \wedge e_j, i<j\}$

$\wedge : T_pM \times T_pM \to T_pM \wedge T_pM$

$(v, w) \mapsto v \wedge w$: bilinear skewsymm.

$B: V \times V \longrightarrow \mathbb{R}$ bilinear, symmetric

$\hat{B}:$ 〃

<u>Fact</u> $B(X, X) = \hat{B}(X, X)$ for $\forall X \in V$

diagonal components

$\Rightarrow B = \hat{B}$

$\because$ $\underline{\hat{B}(X+Y, X+Y)} = B(X, X) + B(X, Y) + B(Y, X) + B(Y, Y)$

$= \underline{\hat{B}(X, X)} + 2\hat{B}(X, Y) + \underline{\hat{B}(Y, Y)}$

$\boldsymbol{K}$: symmetric bilinear form on $T_pM \wedge T_pM$

Definition

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The sectional curvature of (M, g) with respect to the plane Π_p is a number

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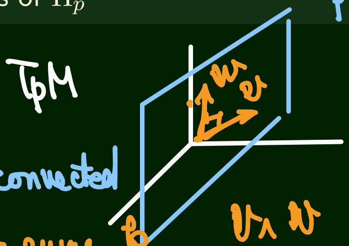
$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

(scalar)

2 dim subsp Π_p



Goal A complete simply connected Riem. mfd of const. sect. curv k is isometric to $H^n(k)$, $\mathbb{R}^n$, $S^n(k)$.

An alternative way to define Sect. curvature.

Lemma

For any function $f \in \mathcal{F}(M)$ and vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = fR(X,Y)Z$$

holds.

$\left(X = \sum X^i \frac{\partial}{\partial x^i} \right)$

Corollary

Assume the vector fields X, Y, Z and $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{X}(M)$ satisfy $X_p = \widetilde{X}_p$, $Y_p = \widetilde{Y}_p$ and $Z_p = \widetilde{Z}_p$ for a point $p \in M$. Then

$$\big(R(X,Y)Z\big)_p = \big(R(\widetilde{X},\widetilde{Y})\widetilde{Z}\big)_p.$$

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

The curvature tensor

Definition

The quadrilinear map

$$R(X, Y, Z, T) = \langle R(X, Y)Z, T\rangle : \mathfrak{X}(M)^4 \to \mathcal{F}(M)$$

is called the curvature tensor.

$R_p : (T_pM)^4 \to \mathbb{R}$

Lemma

$$\kappa_i^j(X, Y) = R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j)$$

$$K^j_i(e_k, e_l) = d\omega^j_i(e_k, e_l) + \sum_m \omega^m_i \wedge \omega^j_m(e_k, e_l)$$

$$= e_k\, \omega^j_i(e_l) - e_l\, \omega^j_i(e_k) - \omega^j_i([e_k, e_l])$$

$$+ \sum_m \omega^m_i(e_k)\, \omega^j_m(e_l) - \omega^m_i(e_l)\, \omega^j_m(e_k)$$

② ---

$$= R(e_k, e_l, e_i, e_j)$$

The curvature tensor

Proposition

- $R(X,Y,Z,T) = -R(Y,X,Z,T) = -R(X,Y,T,Z)$,
- $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$, ← Bianchi
- $R(X,Y,Z,T) = R(Z,T,X,Y)$.

$$\boxed{K(\Pi_p) = \frac{R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x})}{\langle \boldsymbol{x},\boldsymbol{x}\rangle \langle \boldsymbol{y},\boldsymbol{y}\rangle - \langle \boldsymbol{x},\boldsymbol{y}\rangle^2}.}$$

Exercise 5-1

Problem

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \textit{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$\varphi'(r)$

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant* $\underbrace{\textit{sectional curvature}}_{\text{scalar}}$*.*

Exercise 5-2

dim $M = n$ hypersurface

Problem

Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$\boxed{d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j}$$

holds on U.