Advanced Topics in Geometry F1 (MTH.B506)

Sectional Curvature

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Review

- (M, g): a Riemannian n-manifold; $\langle\ ,\ \rangle$: the inner product w. r. to g.
- $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \dots, \omega^n)$: the dual frame of $(\boldsymbol{e}_j)$:

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

- ∇: the Levi-Civita connection on (M, g)
- $\Omega := (\omega_i^j)$: the connection form:

$$\nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega$$

- $\omega_j^k = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form
- $\kappa_i^j = d\omega_i^j + \sum_l \omega_l^j \wedge \omega_i^l$.

Review

- $\omega_j^k = -\omega_k^j$, $\kappa_j^k = -\kappa_k^j$
- $d\omega^i = \sum_{l=1}^{n} \omega^l \wedge \omega_l^i$

cf.

- $X \langle Y, Z \rangle = \langle \nabla_X Y, \nabla Z \rangle + \langle Y, \nabla_X Z \rangle$
- $\nabla_X Y - \nabla_Y X = [X, Y]$

The Bianchi identity

Proposition (The first Bianchi identity; Prop. 5.2)

$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) + \kappa^i_k(\boldsymbol{e}_l, \boldsymbol{e}_j) + \kappa^i_l(\boldsymbol{e}_j, \boldsymbol{e}_k) = 0.$

$\because dd\omega^i = 0.$

The Bianchi identity

Corollary (Cor. 5.3)

$\kappa^i_j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa^k_l(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

The bilinear form derived from the curvature form

$$\boldsymbol{K}(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k,\boldsymbol{e}_l)\xi^{kl}\eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l}\xi^{kl}\boldsymbol{e}_k\wedge\boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j}\eta^{ij}\boldsymbol{e}_i\wedge\boldsymbol{e}_j$$

Lemma

$\boldsymbol{K}$ *is symmetric.*

The sectional curvature

Definition

Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The sectional curvature of (M, g) with respect to the plane Π_p is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of Π_p

The curvature tensor

Lemma

For any function $f \in \mathcal{F}(M)$ *and vector fields* X, Y, $Z \in \mathfrak{X}(M)$,

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z$$

holds.

Corollary

Assume the vector fields X, Y, Z *and* $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \mathfrak{X}(M)$ *satisfy* $X_p = \widetilde{X}_p$, $Y_p = \widetilde{Y}_p$ *and* $Z_p = \widetilde{Z}_p$ *for a point* $p \in M$. *Then*

$$\big(R(X, Y)Z\big)_p = \big(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}\big)_p.$$

The curvature tensor

Definition

The quadrilinear map

$$R(X, Y, Z, T) = \langle R(X, Y)Z, T\rangle : \mathfrak{X}(M)^4 \to \mathcal{F}(M)$$

is called the curvature tensor.

Lemma

$$\kappa_i^j(X, Y) = R(X, Y, \boldsymbol{e}_i, \boldsymbol{e}_j)$$

The curvature tensor

Proposition

- $R(X, Y, Z, T) = -R(Y, X, Z, T) = -R(X, Y, T, Z)$,
- $R(X, Y, Z, T) + R(Y, Z, X, T) + R(Z, X, Y, T) = 0$,
- $R(X, Y, Z, T) = R(Z, T, X, Y)$.

$$K(\Pi_p) = \frac{R(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{y}, \boldsymbol{x})}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{y}, \boldsymbol{y} \rangle - \langle \boldsymbol{x}, \boldsymbol{y} \rangle^2}.$$

Exercise 5-1

Problem

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2 \, d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

Exercise 5-2

Problem

Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U.