Advanced Topics in Geometry F1 (MTH.B506)

Space Forms

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Problem 5-1

Problem

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2 \, d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ *and* φ *is a positive smooth function defined on* $(0, r_0)$ *with*

$$\lim_{r\to+0} \varphi(r) = 0, \qquad \lim_{r\to+0} \frac{\varphi(r)}{r} = 1.$$

k

Classify the function φ *so that* g *is of constant sectional curvature.*

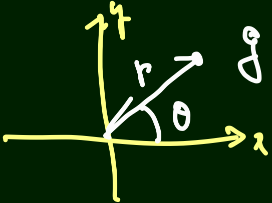
$g = dr^2 + \{\varphi(r)\}^2 d\theta^2$ $\quad e_1 = \frac{\partial}{\partial r}$ $\quad e_2 = \frac{1}{\varphi}\frac{\partial}{\partial \theta}$

$K_1^2 = -\frac{\varphi''}{\varphi}\, \omega^1 \wedge \omega^2$

sectional curvature $= \mathbb{K}(e_1 \wedge e_2, e_1 \wedge e_2)$

$= \boxed{-\frac{\varphi''}{\varphi} = k}$ $\quad \varphi(0) = 0 \quad \varphi'(0) = 1$

$(k = 0)$ $\quad \varphi'' = 0 \quad \varphi(r) = r$

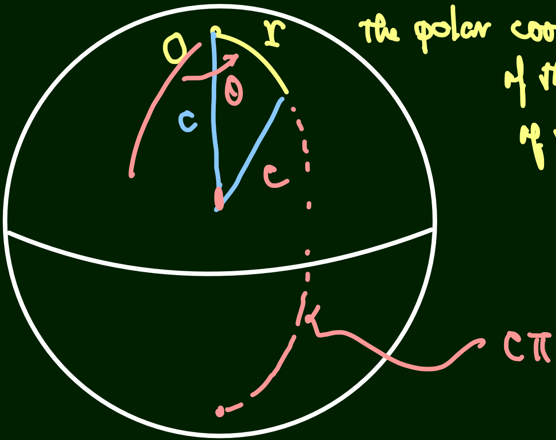


$g = dr^2 + r^2 d\theta^2$: the Euclidean metric with resp. to the polar coord's.

$= dx^2 + dy^2$

$(k = \tfrac{1}{c^2} > 0)$ $\quad \varphi'' = -\tfrac{1}{c^2}\varphi \quad \varphi(0)=0 \quad \varphi'(0)=1$

$\varphi = c\,\cancel{\tfrac{1}{c}} \sin \cancel{c}\,\tfrac{r}{c} > 0 \qquad r \in (0, \cancel{\tfrac{\pi}{c}}\ c\pi)$

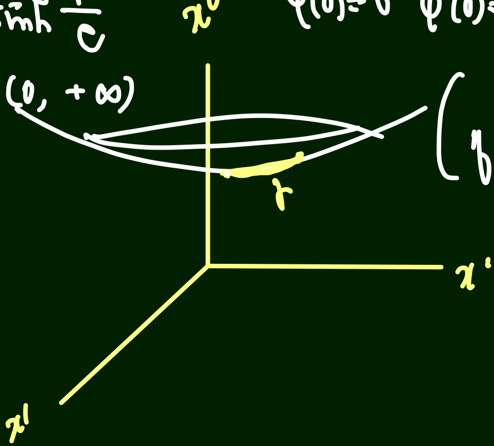


the polar coord's of the sphere of radius c.

$\left(k = -\frac{1}{c^2} < 0 \right)$ $\quad \varphi'' = \frac{\varphi}{c}$ $\quad : \varphi \in \mathrm{Span}\left\{ \cosh \frac{r}{c}, \sinh \frac{r}{c} \right\}$

$\varphi = c \sinh \frac{r}{c}$ $\quad \varphi(0) = 0 \quad \varphi'(0) = 1$

$r \in (0, +\infty)$



(polar coordinates of the hyperboloid.

Problem 5-2

Problem

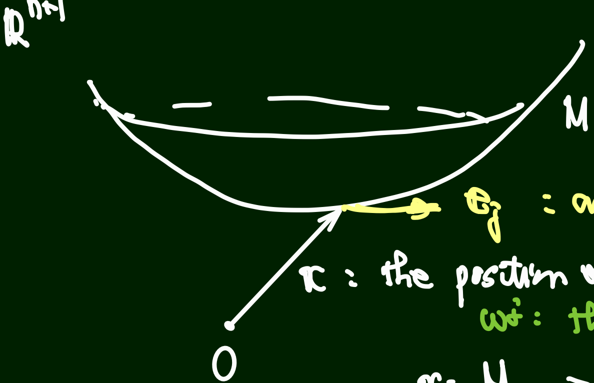
Let $M \subset \mathbb{R}^{n+1}$ *be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of* $\mathbb{R}^{n+1}$*. Then the position vector* $\boldsymbol{x}(p)$ *of* $p \in M$ *induces*

$$\boldsymbol{x} \colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$*-tuple of* C^∞*-functions. Let* $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ *be an orthonormal frame defined on a domain* $U \subset M$*. Since* $T_pM \subset \mathbb{R}^{n+1}$*, we can consider that* $\boldsymbol{e}_j$ *is a smooth map from* $U \to \mathbb{R}^{n+1}$*. Take a dual basis* (ω^j) *to* $[\boldsymbol{e}_j]$*. Prove that*

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U*.*



e_j : an orthonormal frame.

x : the position vector

ω^j : the dual $\left(\omega^j(e_k) = \delta^j_k \right)$

$x : M \to \mathbb{R}^{n+1}$

$$dx = \sum_{j=1}^{n} \omega^j e_j$$

$$\text{at } p \quad dx(e_k) = \frac{d}{dt}\Big|_{t=0} x(\gamma_k(t))$$

$$\left\{ \begin{array}{l} \gamma_k(t) : \text{a curve on } M \\ \gamma_k(0) = p \\ \dot{\gamma}_k(0) = e_k \end{array} \right\}$$

$$= \frac{d}{dt}\Big|_{t=0} \gamma_k(t) = \dot{\gamma}_k(0) = e_k$$

$$dx = \sum \blacksquare_j \, \omega^j . \qquad \blacksquare_k = dx(e_k) = e_k \quad]$$