Advanced Topics in Geometry F1 (MTH.B506)

Space Forms

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2023/07/25

Problem 5-1

Problem

Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\,d\theta^2$$

on $\quad U := \{(r,\theta)\,;\,0 < r < r_0, -\pi < \theta < \pi\},$

where $r_0 \in (0,+\infty]$ *and* φ *is a positive smooth function defined on* $(0,r_0)$ *with*

$$\lim_{r\to+0}\varphi(r) = 0, \qquad \lim_{r\to+0}\frac{\varphi(r)}{r} = 1.$$

Classify the function φ *so that* g *is of constant sectional curvature.*

Problem 5-2

Problem

Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces

$$\boldsymbol{x} \colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j \omega^j$$

holds on U.