Advanced Topics in Geometry F1 (MTH.B506)

Space Forms 空間型

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Review

- (M, g): a Riemannian n-manifold; $\langle \ , \ \rangle$: the inner product w. r. to g.
- $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \ldots, \omega^n)$: the dual frame of $(\boldsymbol{e}_j)$:

1 forms

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

$\omega^j(e_k) = \delta^j_k$

- ∇: the Levi-Civita connection on (M, g).

vector valued

$\nabla_* Y$: 1 form

- $\Omega := (\omega_i^j)$: the connection form:

$$\nabla[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\Omega$$

$\nabla e_j = \sum \omega_j^l e_l$

- $\omega_j^k = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form
- $\kappa_i^j = d\omega_i^j + \sum_l \omega_l^j \wedge \omega_i^l$.

Review

- $\omega_j^k = -\omega_k^j$, $\kappa_j^k = -\kappa_k^j$

Ω . K : skew symmetric .

- $d\omega^i = \sum_{l=1}^{n} \omega^l \wedge \omega_l^i$
- $\kappa_j^i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa_l^k(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

Symmetric bilinear

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl} \boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij} \boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

$\Pi_p = \mathrm{Span}\{v, w\} \subset T_pM$

sectional curvature.

Constant sectional curvature

Theorem

Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as

$$\kappa^i_j = k\omega^i \wedge \omega^j = \qquad \kappa^i_j = \sum_{a<b} \omega^a \wedge \omega^b$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

$$k = K(e_i \wedge e_j, e_i \wedge e_j) = \kappa^i_j(e_i, e_j)$$

Constant sectional curvature

Theorem

$K(\Pi_p) = k$ *for* $\forall \Pi_p \in T_pM \Leftrightarrow \kappa^i_j = k\omega^i \wedge \omega^j$.

$(v \perp w,\ |v| = |w| = 1)$

$$k = \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w})$$

$$\boldsymbol{v} := \cos\theta \boldsymbol{e}_i + \sin\theta \boldsymbol{e}_j, \qquad \boldsymbol{w} := \cos\varphi \boldsymbol{e}_l + \sin\varphi \boldsymbol{e}_m$$

$\Rightarrow$

$$\boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0,$$
$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0$$

$\kappa^i_j(e_k\ \ e_l) = 0$
unless $\{i, j\} = \{k, l\}$

Constant sectional curvature

$n \geqq 3$

$\{$ 2-dim subsp of $T_pM \}$

Theorem

Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p) = k(p)$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \mapsto k(p) \in \mathbb{R}$ is constant provided that M is connected.

Constant sectional curvature

Theorem

$K(\Pi_p) = k(p)$ *for* $\forall \Pi_p \in \mathrm{Gr}_2(T_pM)$, $\forall p \in M \Rightarrow k(p)$ *is constant.*

$d\kappa_i^j = d\left\{ d\omega_i^j - \sum \omega_s^j \wedge \omega_i^s \right\}$

$$d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s \right),$$

$\kappa_i^j = k\,\omega^j \wedge \omega^i$

$d\kappa_i^j$

$\boxed{dk = 0}$

$K(\Pi_p) = \text{const at } p \Rightarrow \kappa^i_j = k\,\omega^i \wedge \omega^j$

$\underset{\substack{\dim \geq 3 \\ \text{connected}}}{\Rightarrow} \quad K(\Pi_p) = k(p) = \text{const. on } M.$

1Q $\forall$ geodesics are defined on $\mathbb{R}$.

Definition 空間型 完備

An n-dimensional space form is a complete Riemannian n-manifold of constant sectional curvature.

(Goal)

M : a space form of curvature k

connected & simply conected.

$$\Rightarrow \quad M = \begin{cases} S^n(k) & (k>0) \\ \mathbb{R}^n = \mathbb{E}^n & (k=0) \\ H^n(k) & (k<0) \end{cases}$$

The Euclidean space

Example

the

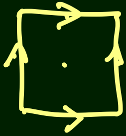
The Euclidean n-space is a simply connected space form of constant curvature 0.

$(K = 0)$

$K=0$

Example

cylinder.



a flat torus

non-simply connected.

The Hyperbolic space

Example

the

The n-dimensional hyperbolic space $H^n(-c^2)$ is ~~a~~ simply connected space form of constant curvature $-c^2$.

Example

The ~~3~~ 2-dimensional hyperbolic space $H^{\cancel{3}\,2}(-\cancel{c^2}\,1)$ is a simply connected space form of constant curvature $-\cancel{c^2}\,1$.

$$H^2(-1) := \{ x = (x^0, x^1, x^2) \in \mathbb{R}^3_1 \,,\ \langle x, x \rangle = -1 \,,\ x^0 > 0 \}$$

$$T_x H^2(-1) = x^{\perp} = \{ v \mid \langle x, v \rangle = 0 \}$$

Set $e_0 := x$ ⊥, $\{e_1, e_2\}$ an orthonormal frame

$\mathcal{F} := (e_0 \quad e_1 \quad e_2)$ $\quad {}^t\mathcal{F} Y \mathcal{F} = Y = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$e_0 = x$ orthonormal $\Leftrightarrow$

ω^1, ω^2: the dual frame to $\{e_1, e_2\}$

ω^i_j: the connection

$\omega^1_2 = -\omega^2_1$

Set $d\mathcal{F} = \mathcal{F}\widetilde{\Omega}$.

- Compute $\widetilde{\Omega}$:

$de_0 = dx = \omega^1 e_1 + \omega^2 e_2$

$[de_1]^T = \nabla e_1 = \omega^2_1 e_2 \quad [de_2]^T = \omega^1_2 e_1$ ← tangent component

$\langle de_1, e_0 \rangle = \cancel{d\langle e_1, e_0\rangle} - \langle e_1, de_0\rangle$

$\langle e_1, e_0\rangle = 0$ $\quad - \langle e_1, \omega^1 e_1 + \omega^2 e_2\rangle = -\omega^1$

$\langle de_2, e_0 \rangle = -\omega^2$

$$dF = F\,\widehat{\Omega}$$

$$d\theta_0 = \omega^1 \theta_1 + \omega^2 \theta_2$$

$$d\theta_1 = \omega^1 \theta_0 + \omega_1^2 \theta_2$$

$$d\theta_2 = \omega^2 \theta_0 + \omega_2^1 \theta_1$$

$$\widehat{\Omega} = \begin{pmatrix} 0 & \omega^1 & \omega^2 \\ \omega^1 & 0 & \omega_1^2 \\ \omega^2 & \omega_2^1 & 0 \end{pmatrix} \qquad \Omega$$

$$= \begin{pmatrix} 0 & {}^t\omega \\ \omega & \Omega \end{pmatrix}$$

Integrability : $d\widehat{\Omega} + \widehat{\Omega} \wedge \widehat{\Omega} = 0$

$$\hat{\Omega} = \begin{pmatrix} 0 & \omega^T \\ \omega & \Omega \end{pmatrix}$$

$$d\hat{\Omega} + \hat{\Omega}\wedge\hat{\Omega} = \begin{pmatrix} \omega^T\wedge\omega & \cancel{d\omega^T + \omega^T\wedge\Omega} \\ \cancel{d\omega + \Omega\wedge\omega} & \boxed{d\Omega + \Omega\wedge\Omega + \omega\wedge\omega^T} \end{pmatrix}$$

$$(\omega^1, \omega^2)\wedge\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = \omega^1\wedge\omega^1 + \omega^2\wedge\omega^2 = 0$$

$d\omega + \Omega\wedge\omega = 0$ because $d\omega^i = \sum \omega^l \wedge \omega_l^i$

By integrability,

$$d\Omega + \Omega\wedge\Omega = -\omega\wedge\omega^T = -\begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}\wedge(\omega^1, \omega^2)$$

$$K = -\begin{pmatrix} \omega^1 \wedge \omega^1 & \omega^2 \wedge \omega^1 \\ \omega^1 \wedge \omega^2 & \omega^2 \wedge \omega^2 \end{pmatrix} = -\begin{pmatrix} 0 & \omega^2 \wedge \omega^1 \\ \underline{\omega^1 \wedge \omega^2} & 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\text{sec. curv} = -1}$$

The Main Theorem

Theorem

Let M be a simply connected n-manifold and g a Riemannian metric on M. If the sectional curvature of (M, g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

Local uniqueness theorem

Theorem

Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U, g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$.

Isometry

Definition

A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M, g) and (N, h) is called a local isometry if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X, Y) := h(df(X), df(Y)) = g(X, Y)$$

holds for X, $Y \in T_pM$ and $p \in M$.

Fact (Corollary 6.10)

A smooth map $f\colon (M, g) \to (N, h)$ is a local isometry if and only if for each $p \in M$,

$$[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] := [df(\boldsymbol{e}_1), \dots, df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a

The Special Case

Theorem

Let $U \subset \mathbb{R}^2$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (M, g) is constant -1, there exists a local isometry $f \colon U \to H^2(-1)$.

Solve $\underline{dF = F\tilde{\Omega}}$ with $F(p) = \mathrm{id}$.

determined by (U, g)

$$\tilde{\Omega} = \begin{pmatrix} 0 & \omega^1 & \omega^2 \\ \omega^1 & 0 & \omega_1^2 \\ \omega^2 & \omega_2^1 & 0 \end{pmatrix}$$

possibility $\Leftrightarrow$ integrability $\Leftrightarrow$ const. curv -1

f = the 1st column of F.

Exercise 6-1

Problem

Prove that the sphere

$$S^3 = \{\boldsymbol{x} \in \mathbb{R}^4 ; \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

of radius 1 *in the Euclidean* 4*-space is of constant sectional curvature* 1*.*

Exercise 6-2

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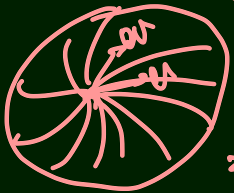
Problem 6.11 6-1

Prove Theorem ~~??~~ for $k = 1$ and $n = 2$ assuming Exercise ~~??~~.

- Sec curv.

M

- (Ricci curvature
- Scalar curvature)



Σ^2

$\downarrow$

Gauss. curv.

= sec curv