Advanced Topics in Geometry F1 (MTH.B506)

Space Forms

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Review

- (M, g): a Riemannian n-manifold; $\langle\ ,\ \rangle$: the inner product w. r. to g.
- $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$: an orthonormal frame on $U \subset M$.
- $(\omega^1, \ldots, \omega^n)$: the dual frame of $(\boldsymbol{e}_j)$:

$$\omega^j = \langle \boldsymbol{e}_j, * \rangle$$

- ∇: the Levi-Civita connection on (M, g)
- $\Omega := (\omega_i^j)$: the connection form:

$$\nabla[\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n] = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n]\Omega$$

- $\omega_j^k = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k \rangle$
- $K = (\kappa_i^j) = d\Omega + \Omega \wedge \Omega$: the curvature form
- $\kappa_i^j = d\omega_i^j + \sum_l \omega_l^j \wedge \omega_i^l$.

Review

- $\omega_j^k = -\omega_k^j$, $\kappa_j^k = -\kappa_k^j$
- $d\omega^i = \sum_{l=1}^{n} \omega^l \wedge \omega_l^i$
- $\kappa_j^i(\boldsymbol{e}_k, \boldsymbol{e}_l) = \kappa_l^k(\boldsymbol{e}_i, \boldsymbol{e}_j)$.

$$\boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij},$$

$$\boldsymbol{\xi} = \sum_{k<l} \xi^{kl}\boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

Constant sectional curvature

Theorem

Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as

$$\kappa^i_j = k\omega^i \wedge \omega^j.$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

Constant sectional curvature

Theorem

$K(\Pi_p) = k$ *for* $\forall \Pi_p \in T_pM \Leftrightarrow \kappa^i_j = k\omega^i \wedge \omega^j$.

$$k = \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w})$$

$$\boldsymbol{v} := \cos\theta \boldsymbol{e}_i + \sin\theta \boldsymbol{e}_j, \qquad \boldsymbol{w} := \cos\varphi \boldsymbol{e}_l + \sin\varphi \boldsymbol{e}_m$$

$\Rightarrow$

$$\boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0,$$
$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0$$

Constant sectional curvature

Theorem

Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p) = k(p)$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \to k(p) \in \mathbb{R}$ is constant provided that M is connected.

Constant sectional curvature

Theorem

$K(\Pi_p) = k(p)$ *for* $\forall \Pi_p \in \mathrm{Gr}_2(T_pM)$, $\forall p \in M \Rightarrow k(p)$ *is constant.*

$$d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s \right),$$

Space forms

Definition

An n-dimensional space form is a complete Riemannian n-manifold of constant sectional curvature.

The Euclidean space

Example

The Euclidean n-space is a simply connected space form of constant curvature 0.

The Hyperbolic space

Example

The n-dimensional hyperbolic space $H^n(-c^2)$ is a simply connected space form of constant curvature $-c^2$.

The Hyperbolic 3-space

Example

The 3-dimensional hyperbolic space $H^3(-c^2)$ is a simply connected space form of constant curvature $-c^2$.

The Main Theorem

Theorem

Let M be a simply connected n-manifold and g a Riemannian metric on M. If the sectional curvature of (M, g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

Local uniqueness theorem

Theorem

Let $U \subset \mathbb{R}^n$ *be a simply connected domain and* g *a Riemannian metric on* U*. If the sectional curvature of* (U, g) *is constant* k*, there exists a local isometry* $f \colon U \to N^n(k)$*.*

Isometry

Definition

A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M, g) and (N, h) is called a local isometry if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X, Y) := h(df(X), df(Y)) = g(X, Y)$$

holds for X, $Y \in T_pM$ and $p \in M$.

Fact (Corollary 6.10)

A smooth map $f\colon (M, g) \to (N, h)$ is a local isometry if and only if for each $p \in M$,

$$[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] := [df(\boldsymbol{e}_1), \ldots, df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

The Special Case

Theorem

Let $U \subset \mathbb{R}^2$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (M, g) is constant -1, there exists a local isometry $f \colon U \to H^2(-1)$.

Exercise 6-1

Problem

Prove that the sphere

$$S^3 = \{\boldsymbol{x} \in \mathbb{R}^4\,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

of radius 1 *in the Euclidean* 4*-space is of constant sectional curvature* 1*.*

Exercise 6-2

Problem

Prove Theorem **??** *for* $k = 1$ *and* $n = 2$*, assuming Exercise* **??**.