# 1 Linear Ordinary Differential Equations

**The fundamental theorem for ordinary differential equations.** Consider a function

$$\boldsymbol{f}\colon I\times U \ni (t,\boldsymbol{x}) \longmapsto \boldsymbol{f}(t,\boldsymbol{x}) \in \mathbb{R}^m \tag{1.1}$$

of class $C^1$, where $I \subset \mathbb{R}$ is an interval and $U \subset \mathbb{R}^m$ is a domain in the Euclidean space $\mathbb{R}^m$. For any fixed $t_0 \in I$ and $\boldsymbol{x}_0 \in U$, the condition

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}\bigl(t,\boldsymbol{x}(t)\bigr), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1.2}$$

of an $\mathbb{R}^m$-valued function $t \mapsto \boldsymbol{x}(t)$ is called the *initial value problem of ordinary differential equation* for unknown function $\boldsymbol{x}(t)$. A function $\boldsymbol{x}\colon I \to U$ satisfying (1.2) is called a *solution* of the initial value problem.

**Fact 1.1** (The existence theorem for ODE's)**.** *Let $\boldsymbol{f}\colon I\times U \to \mathbb{R}^m$ be a $C^1$-function as in* (1.1)*. Then, for any $\boldsymbol{x}_0 \in U$ and $t_0 \in I$, there exists a positive number $\varepsilon$ and a $C^1$-function $\boldsymbol{x}\colon I \cap (t_0 - \varepsilon, t_0+\varepsilon) \to U$ satisfying* (1.2)*.*

Consider two solutions $\boldsymbol{x}_j\colon J_j \to U$ $(j=1,2)$ of (1.2) defined on subintervals $J_j \subset I$ containing $t_0$. Then the function $\boldsymbol{x}_2$ is said to be an *extension* of $\boldsymbol{x}_1$ if $J_1 \subset J_2$ and $\boldsymbol{x}_2|_{J_1} = \boldsymbol{x}_1$. A solution $\boldsymbol{x}$ of (1.2) is said to be *maximal* if there are no non-trivial extension of it.

**Fact 1.2** (The uniqueness for ODE's)**.** *The maximal solution of* (1.2) *is unique.*

**Fact 1.3** (Smoothness of the solutoins)**.** *If $\boldsymbol{f}\colon I\times U \to \mathbb{R}^m$ is of class $C^r$ $(r=1,\dots,\infty)$, the solution of* (1.2) *is of class $C^{r+1}$. Here, $\infty+1=\infty$, as a convention.*

Let $V \subset \mathbb{R}^k$ be another domain of $\mathbb{R}^k$ and consider a $C^\infty$-function

$$\boldsymbol{h}\colon I\times U\times V \ni (t,\boldsymbol{x};\boldsymbol{\alpha}) \mapsto \boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha}) \in \mathbb{R}^m. \tag{1.3}$$

For fixed $t_0 \in I$, we denote by $\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ the (unique, maximal) solution of (1.2) for $\boldsymbol{f}(t,\boldsymbol{x}) = \boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})$. Then

**Fact 1.4.** *The map $(t,\boldsymbol{x}_0;\boldsymbol{\alpha}) \mapsto \boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ is of class $C^\infty$.*

**Example 1.5.** (1) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=\lambda x$, where $\lambda$ is a constant. Then $x(t) = x_0\exp(\lambda t)$ defined on $\mathbb{R}$ is the maximal solution to

$$\frac{d}{dt}x(t) = f(t,x(t)) = \lambda x(t), \qquad x(0)=x_0.$$

(2) Let $m=2$, $I=\mathbb{R}$, $U=\mathbb{R}^2$ and $\boldsymbol{f}(t;(x,y)) = (y,-\omega^2 x)$, where $\omega$ is a constant. Then

$$\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} x_0\cos\omega t + \frac{y_0}{\omega}\sin\omega t\\ -x_0\omega\sin\omega t + y_0\cos\omega t\end{pmatrix}$$

is the unique solution of

$$\frac{d}{dt}\begin{pmatrix} x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix} y(t)\\ -\omega^2 x(t)\end{pmatrix}, \qquad \begin{pmatrix} x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix} x_0\\ y_0\end{pmatrix},$$

defined on $\mathbb{R}$. This differential equation can be considered a single equation

$$\frac{d^2}{dt^2}x(t) = -\omega^2 x(t), \quad x(0)=x_0, \quad \frac{dx}{dt}(0) = y_0$$

of order 2.

(3) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=1+x^2$. Then $x(t)=\tan t$ defined on $(-\frac{\pi}{2},\frac{\pi}{2})$ is the unique maximal solution of the initial value problem

$$\frac{dx}{dt} = 1+x^2, \qquad x(0)=0.$$

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**Linear Ordinary Differential Equations.** The ordinary differential equation (1.2) is said to be *linear* if the function (1.1) is a linear function in $\boldsymbol{x}$, that is, a linear differential equation is in a form

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

where $A(t)$ and $\boldsymbol{b}(t)$ are $m \times m$-matrix-valued and $\mathbb{R}^m$-valued functions in $t$.

For the sake of later use, we consider, in this lecture, the special form of linear differential equation for matrix-valued unknown functions as follows: Let $\mathrm{M}_n(\mathbb{R})$ be the set of $n \times n$-matrices with real components, and take functions

$$\Omega\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}), \quad \text{and} B\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}),$$

where $I \subset \mathbb{R}$ is an interval. Identifying $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, we assume $\Omega$ and $B$ are continuous functions (with respect to the topology of $\mathbb{R}^{n^2} = \mathrm{M}_n(\mathbb{R})$). Then we can consider the linear ordinary differential equation for matrix-valued unknown $X(t)$ as

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0, \tag{1.4}$$

where $X_0$ is given constant matrix.

Then, the fundamental theorem of *linear* ordinary equation states that *the maximal solution of* (1.4) *is defined on whole* $I$. To prove this, we prepare some materials related to matrix-valued functions.

**Preliminaries: Matrix Norms.** Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$-matrices with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}} = \sqrt{\operatorname{tr}(X^T X)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2} \tag{1.5}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\, \boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\}, \tag{1.6}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

**Lemma 1.6.** (1) *The map* $X \mapsto |X|_{\mathrm{M}}$ *is a norm of* $\mathrm{M}_n(\mathbb{R})$.

(2) *For* $X$*,* $Y \in \mathrm{M}_n(\mathbb{R})$*, it holds that* $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.

(3) *Let* $\lambda = \lambda(X)$ *be the maximum eigenvalue of semi-positive definite symmetric matrix* $X^T X$. *Then* $|X|_{\mathrm{M}} = \sqrt{\lambda}$ *holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map* $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ *is continuous with respect to the Euclidean norm.*

*Proof.* Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v} \in S^{n-1}\}$, where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (1.6) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm[1].

[1] $|X|_{\mathrm{M}} > 0$ whenever $X \neq O$, $|\alpha X|_{\mathrm{M}} = |\alpha|\,|X|_{\mathrm{M}}$, and the triangle inequality.

Since $A := X^T X$ is positive semi-definite, the eigenvalues $\lambda_j$ $(j = 1, \ldots, n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j$ $(j = 1, \ldots, n)$. Let $\lambda$ be the maximum eigenvalue of $A$, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \cdots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \cdots + \lambda_n v_n^2 \leqq \lambda \langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the $\lambda$-eigenvector, proving (3). Noticing the norm (1.5) is invariant under conjugations $X \mapsto P^T X P$ $(P \in \mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$ by diagonalizing $X^T X$ by an orthogonal matrix $P$. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

**Preliminaries: Matrix-valued Functions.**

**Lemma 1.7.** *Let $X$ and $Y$ be $C^\infty$-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

*where $\widetilde{X}$ is the cofactor matrix of $X$, and we assume in (3) that $X$ is a regular matrix.*

*Proof.* The formula (1) holds because the definition of matrix multiplication and the Leibnitz rule, Denoting $' = \partial/\partial u_j$,

$$O = (\mathrm{id})' = (X^{-1}X)' = (X^{-1})X' + (X^{-1})'X$$

implies (3), where id is the identity matrix.

Decompose the matrix $X$ into column vectors as $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$. Since the determinant is multi-linear form for $n$-tuple of column vectors, it holds that

$$(\det X)' = \det(\boldsymbol{x}_1', \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) + \det(\boldsymbol{x}_1, \boldsymbol{x}_2', \ldots, \boldsymbol{x}_n) + \cdots + \det(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n').$$

Then by cofactor expansion of the right-hand side, we obtain (2). □

**Proposition 1.8.** *Assume two $C^\infty$ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.7}$$

*Then*

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\Omega(\tau)\,d\tau \tag{1.8}$$

*holds. In particular, if $X_0 \in \mathrm{GL}(n, \mathbb{R})$,*[2] *then $X(t) \in \mathrm{GL}(n, \mathbb{R})$ for all $t$.*

*Proof.* By (2) of Lemma 1.7, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[3]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.8). □

[2] $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.

[3] In this lecture, id denotes the identity matrix.

**Corollary 1.9.** *If $\Omega(t)$ in (1.7) satisfies $\operatorname{tr}\Omega(t) = 0$, $\det X(t)$ is constant. In particular, if $X_0 \in \mathrm{SL}(n,\mathbb{R})$, $X$ is a function valued in $\mathrm{SL}(n,\mathbb{R})$* [4].

**Proposition 1.10.** *Assume $\Omega(t)$ in (1.7) is skew-symmetric for all $t$, that is, $\Omega^T + \Omega$ is identically $O$. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[5]*, then $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all $t$.*

*Proof.* By (1) in Lemma 1.7,

$$\begin{aligned}\frac{d}{dt}(XX^T) &= \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T \\ &= X\Omega X^T + X\Omega^T X^T = X(\Omega + \Omega^T)X^T = O.\end{aligned}$$

Hence $XX^T$ is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)X(t)^T = X(t_0)X(t_0)^T = X_0X_0^T = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

**Preliminaries: Norms of Matrix-Valued functions.** Let $I = [a, b]$ be a closed interval, and denote by $C^0(I, \mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X\colon I \to \mathrm{M}_n(\mathbb{R})$. For any positive number $k$, we define

$$||X||_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{1.9}$$

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $||\cdot||_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

**Lemma 1.11.** *The norm $||\cdot||_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

**Linear Ordinary Differential Equations.** We prove the fundamental theorem for *linear* ordinary differential equations.

**Proposition 1.12.** *Let $\Omega(t)$ be a $C^\infty$-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval $I$. Then for each $t_0 \in I$, there exists the unique matrix-valued $C^\infty$-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{1.10}$$

*Proof.* Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.10). Then

$$Y(t) - X(t) = \int_{t_0}^{t}\bigl(Y'(\tau) - X'(\tau)\bigr)\,d\tau = \int_{t_0}^{t}\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}' = \frac{d}{dt}\right)$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned}|Y(t) - X(t)|_{\mathrm{M}} &\leqq \left|\int_{t_0}^{t}\left|\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}} d\tau\right| \leqq \left|\int_{t_0}^{t}|Y(\tau) - X(\tau)|_{\mathrm{M}}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \\ &= \left|\int_{t_0}^{t} e^{-k\tau}\,|Y(\tau) - X(\tau)|_{\mathrm{M}}\,e^{k\tau}\,|\Omega(\tau)|_{\mathrm{M}}\,d\tau\right| \leqq ||Y - X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\left|\int_{t_0}^{t}e^{k\tau}\,d\tau\right| \\ &= ||Y - X||_{J,k}\frac{\sup_J|\Omega|_{\mathrm{M}}}{|k|}e^{kt}\left|1 - e^{-k(t-t_0)}\right| \leqq ||Y - X||_{J,k}\sup_J|\Omega|_{\mathrm{M}}\frac{e^{kt}}{|k|}\end{aligned}$$

[4] $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, \det A = 1\}$; the special lienar group.

[5] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^TA = AA^T = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\, \det A = 1\}$: the special orthogonal group.

holds for $t \in J$. Thus, for an appropriate choice of $k \in \mathbb{R}$, it holds that

$$||Y - X||_{J,k} \leqq \frac{1}{2}||Y - X||_{J,k},$$

that is, $||Y - X||_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Since $J$ is arbitrary, $Y = X$ holds on $I$.
Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on $I$ satisfying $X_0(t) = \mathrm{id}$ and

$$X_{j+1}(t) = \mathrm{id} + \int_{t_0}^{t} X_j(\tau)\Omega(\tau)\,d\tau \quad (j = 0, 1, 2, \dots). \tag{1.11}$$

Let $k := 2\sup_J |\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned}|X_{j+1}(t) - X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^{t} |X_j(\tau) - X_{j-1}(\tau)|_{\mathrm{M}} |\Omega(\tau)|_{\mathrm{M}}\,d\tau \\ &\leqq \frac{e^{k(t-t_0)}}{|k|} \sup_J |\Omega|_{\mathrm{M}} ||X_j - X_{j-1}||_{J,k}\end{aligned}$$

for an appropriate choice of $k \in \mathbb{R}$, and hence $||X_{j+1} - X_j||_{J,k} \leqq \frac{1}{2}||X_j - X_{j-1}||_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $||\cdot||_{J,k}$. Thus, by completeness (Lemma 1.11), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.11), the limit $X$ satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that $X$ satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since $J$ can be taken arbitrarily, existence of the solution on $I$ is proven.

Finally, we shall prove that $X$ is of class $C^\infty$. Since $X'(t) = X(t)\Omega(t)$, the derivative $X'$ of $X$ is continuous. Hence $X$ is of class $C^1$, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class $C^1$, and then $X$ is of class $C^2$. Iterating this argument, we can prove that $X(t)$ is of class $C^r$ for arbitrary $r$. □

**Corollary 1.13.** *Let $\Omega(t)$ be a matrix-valued $C^\infty$-function defined on an interval $I$. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued $C^\infty$-function $X_{t_0,X_0}(t)$ defined on $I$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \bigl(X(t) := X_{t_0,X_0}(t)\bigr) \tag{1.12}$$

*In particular, $X_{t_0,X_0}(t)$ is of class $C^\infty$ in $X_0$ and $t$.*

*Proof.* We rewrite $X(t)$ in Proposition 1.12 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.13}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all $t$ because of Proposition 1.8,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\frac{dW}{dt} = \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1} = X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.13). □

**Proposition 1.14.** *Let $\Omega(t)$ and $B(t)$ be matrix-valued $C^\infty$-functions defined on $I$. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued $C^\infty$-function defined on $I$ satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.14}$$

*Proof.* Rewrite $X$ in Proposition 1.12 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) = \left(X_0 + \int_{t_0}^t B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.15}$$

satisfies (1.14). Conversely, if $X$ satisfies (1.14), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^t B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.15). □

**Theorem 1.15.** *Let $I$ and $U$ be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued $C^\infty$-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued $C^\infty$-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on $I$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1.16}$$

*Moreover,*

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

*is a $C^\infty$-map.*

*Proof.* Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.16) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{1.17}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0,\alpha_1,\dots,\alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{\mathrm{id},X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.17) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.14. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{1.18}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$. In particular, by the proof of Proposition 1.14, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right)Y^{-1}(\tau)d\tau\right)Y(t).$$

Here, $Y(t)$ is the unique matrix-valued $C^\infty$-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a $C^\infty$-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

**Fundamental Theorem for Space Curves.** As an application, we prove the fundamental theorem for space curves. A $C^\infty$-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on $I$. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter $s$ is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all $s$. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ of $\mathbb{R}^3$ for each $s$. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.19}$$

in $s$, which is called the *Frenet frame* associated to the curve $\gamma$. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of $\gamma$. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\varOmega, \qquad \varOmega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.20}$$

**Proposition 1.16.** *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

*Proof.* Let $\kappa$, $\tau$ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of $\gamma_1$, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.20), and then $\gamma_1$ and $\gamma_2$ have common curvature and torsion.

Conversely, assume $\gamma_1$ and $\gamma_2$ have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.20). Let $\mathcal{F}$ be the unique solution of (1.20) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.13, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ $(j = 1, 2)$. In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

**Theorem 1.17** (The fundamental theorem for space curves)**.**
*Let $\kappa(s)$ and $\tau(s)$ be $C^\infty$-fnctions defined on an interval $I$ satisfying $\kappa(s) > 0$ on $I$. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are $\kappa$ and $\tau$, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.*

*Proof.* We have already shown the uniqueness in Proposition 1.16. We shall prove the existence: Let $\varOmega(s)$ be as in (1.20), and $\mathcal{F}(s)$ the solution of (1.20) with $\mathcal{F}(s_0) = \mathrm{id}$. Since $\varOmega$ is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.10. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of $\gamma$, and $\kappa$, and $\tau$ are the curvature and torsion of $\gamma$, respectively. □

***Exercises***

**1-1** Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where $b$ is a real number.

**1-2** Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length $s$, whose curvature $\kappa$ and torsion $\tau$ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$