3 Differential Forms

Let M be an n-dimensional manifold and denote by $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ the set of smooth function and the set of smooth vector fields on M, respectively.

Lie brackets A vector field $X \in \mathfrak{X}(M)$ can be considered as a differential operator acting on $\mathcal{F}(M)$ as $(Xf)(\mathrm{P}) = X_{\mathrm{P}} f$. By definition it satisfies the Leibniz rule

$$X(fg) = f(Xg) + g(Xf) \qquad (X \in \mathfrak{X}(M), f, g \in \mathcal{F}(M)). \tag{3.1}$$

For two vector fields X, $Y \in \mathfrak{X}(M)$, set

$$[X, Y] \colon \mathcal{F}(M) \ni f \longmapsto X(Yf) - Y(Xf) \in \mathcal{F}(M). \tag{3.2}$$

Then $[X, Y]$ also satisfies the Leibnitz rule (3.1), and gives a vector field on M. The map

$$[\ ,\ ] \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto [X, Y] \in \mathfrak{X}(M)$$

is called the *Lie bracket* on $\mathfrak{X}(M)$. One can easily show that the product $[\ ,\ ]$ is bilinear, skew symmetric and satisfies the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = \mathbf{0}, \tag{3.3}$$

that is, $(\mathfrak{X}(M), [\ ,\ ])$ is a *Lie algebra* (of infinite dimension). By the Leibniz rule, it holds that

$$[fX, Y] = f[X, Y] - (Yf)X, \quad [X, fY] = f[X, Y] + (Xf)Y \qquad (X, Y \in \mathfrak{X}(M), f \in \mathcal{F}(M)). \tag{3.4}$$

Tensors. For each $\mathrm{p} \in M$, the *dual space* T_p^*M of T_pM is the liner space consisting of all linear maps from T_pM to $\mathbb{R}$.

Lemma 3.1. *Let $(x^1, \dots, x^n)$ be a local coordinate system of M around p, and set*

$$\left(\frac{\partial}{\partial} x^j\right)_p \colon \mathcal{F}(M) \ni f \mapsto \frac{\partial f}{\partial x^j}(p), \qquad (dx^j)_p \colon T_pM \to \mathbb{R} \qquad \text{with} \qquad (dx^j)_p\left(\left(\frac{\partial}{\partial x^k}\right)_p\right) = \delta^j_k$$

for $j, k = 1, \dots, n$. Then $\{(\partial/\partial x^j)_p\}_{j=1,\dots,n}$ and $\{(dx^j)_p\}_{j=1,\dots,n}$ are a basis of T_pM and T_p^*M, respectively, where δ^j_k denotes Kronecker's delta symbol.

We let

$$T_p^*M \otimes T_p^*M \qquad \bigl(\text{resp.} \qquad T_p^*M \otimes T_p^*M \otimes T_p^*M \colon T_pM\bigr)$$

the set of bilinear (resp. trilinear) maps of $T_pM \times T_pM$ (resp. $T_pM \times T_pM \times T_pM$) to $\mathbb{R}$. A section of the vector bundle

$$T^*M \otimes T^*M := \bigcup_{p \in M} T_p^*M \otimes T_p^*M \qquad \left(\text{resp. } T^*M \otimes T^*M \otimes T^*M := \bigcup_{p \in M} T_p^*M \otimes T_p^*M \otimes T_p^*M\right)$$

is called a *covariant* 2 (resp. 3)-*tensor*.

A section $\omega \in \Gamma(T^*M)$ of the cotangent bundle T^*M is called a *covariant* 1-*tensor* or a 1-*form*. A one form ω induces a linear map

$$\omega \colon \mathfrak{X}(M) \ni X \longmapsto \omega(X) \in \mathcal{F}(M), \qquad \text{where} \qquad \omega(X)(p) = \omega_p(X_p) \tag{3.5}$$

By definition, it holds that

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M)). \tag{3.6}$$

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Lemma 3.2. *A linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ is a 1-form if and only if* (3.6) *holds.*

Proof. The "only if" part is trivial by definition. Assume a linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ satisfies (3.6). In fact, under a local coordinate system $(x^1, \dots, x^n)$ around $p \in M$,

$$\omega(X)(p) = \omega\left(\sum_{j=1}^n X^j \frac{\partial}{\partial x^j}\right)(p) = \sum_{j=1}^n X^j(p)\omega\left(\frac{\partial}{\partial x^j}\right)_p, \qquad \left(X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}.\right)$$

holds. In other words, $\omega(X)(p)$ depend only on X_p. Hence ω induces a map $\omega_p\colon T_pM \to \mathbb{R}$. □

Similarly, a *covariant* 2 (resp. 3) tensor $\alpha \in \Gamma(T^*M \otimes T^*M)$ (resp. $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$)induces a bilinear (resp. trilinear) map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$. (resp. $\beta\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$. By the same reason as Lemma 3.2, we have

Lemma 3.3. *A bilinear map $\alpha\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathcal{F}(M)$ (resp. $\beta\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathcal{F}(M)$) is a a covariant 2 (resp. 3)-tensor if and only if*

$$\alpha(fX,Y) = \alpha(X,fY) = f\alpha(X,Y)$$
$$\bigl(\textit{resp.}\quad \beta(fX,Y,Z) = \beta(X,fY,Z) = \beta(X,Y,fZ) = f\beta(X,Y,Z)\bigr)$$

holds for all X, Y, $Z \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$.

A covariant 2 (resp. 3)-tensor α (resp. β) said to be *skew-symmetric* if

$$\alpha(X,Y) = -\alpha(Y,X), \quad \bigl(\beta(X,Y,Z) = -\beta(Y,X,Z) = -\beta(X,Z,Y) = -\beta(Z,Y,X)\bigr)$$

holds for all X, Y, $Z \in \mathfrak{X}(M)$. We denote

$$\wedge^k(M) := \begin{cases} \mathcal{F}(M & (k=0),\\ \Gamma(T^*M) & (k=1),\\ \bigl\{\omega \in \Gamma\bigl(T^*M \otimes T^*M\bigr)\,;\, \omega \text{ is skew-symmetric}\bigr\} & (k=2),\\ \bigl\{\omega \in \Gamma\bigl(T^*M \otimes T^*M \otimes T^*M\bigr)\,;\, \omega \text{ is skew-symmetric}\bigr\} & (k=3). \end{cases} \tag{3.7}$$

An element of $\wedge^k(M)$ is called an *k-form*.

The Exterior products. The *exterior product* $\alpha \wedge \beta \in \wedge^2(M)$ of two 1-forms α, $\beta \in \wedge^1(M)$ is defined as

$$(\alpha \wedge \beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{3.8}$$

On the other hand, the exterior product of α and ω is defined as a 3-form on M by

$$(\alpha \wedge \omega)(X,Y,Z) = (\omega \wedge \alpha)(X,Y,Z) := \alpha(X,Y)\omega(Z) + \alpha(Y,Z)\omega(X) + \alpha(Z,X)\omega(Y). \tag{3.9}$$

Then by a direct computation together with (3.8), it holds that

$$(\mu \wedge \omega) \wedge \lambda = \mu \wedge (\omega \wedge \lambda)\Bigl(=: \mu \wedge \omega \wedge \lambda\Bigr) \tag{3.10}$$

for 1-forms μ, ω and λ.

The Exterior derivative. Under a local coordinate system $(x^1,\dots,x^n)$, a one form α and a two form ω are expressed as

$$\alpha = \sum_{j=1}^{n} \alpha_j\, dx^j, \qquad \omega = \sum_{1\leqq i<j\leqq n} \omega_{ij}\, dx^i \wedge dx^j,$$

where α_j $(j=1,\dots,n)$ and ω_{ij} $(1 \leqq i < j \leqq n)$ are smooth functions in $(x^1,\dots,x^n)$. By Lemma 3.3 and the property (3.4) of the Lie brackets, we have

Lemma 3.4. *For a function $f \in \mathcal{F}(M) = \wedge^0(M)$, a 1-form $\alpha \in \wedge^1(M)$ and a 2-form $\beta \in \wedge^2(M)$)*

$$\begin{aligned}
df &\colon \mathfrak{X}(M) \ni X \mapsto df(X) = Xf \in \mathcal{F}(M),\\
d\alpha &\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y) \mapsto X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]) \in \mathcal{F}(M)\\
d\beta &\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y,Z) \mapsto\\
&\quad X\beta(Y,Z) + Y\beta(Z,X) + Z\beta(X,Y) - \beta([X,Y],Z) - \beta([Y,Z],Z) - \beta([Z,X],Y)
\end{aligned}$$

are a 1-form, a 2-form and a 3-form respectively.

Definition 3.5. For a function f, a 1-form α and a 2-form β, df, $d\alpha$ and $d\beta$ are called the *exterior derivatives* of f, α and β, respectively.

Then, for one forms μ and ω, we have

$$dd\omega = 0, \qquad d(\mu\wedge\omega) = d\mu\wedge\omega - \mu\wedge d\omega, \tag{3.11}$$

by the definition and the Jacobi identity (3.3).

The Riemannian connection. In the rest of this section, we let (M,g) be an n-dimensional (pseudo) Riemannian manifold, and denote by $\langle\ ,\ \rangle$ the inner product induced by g.

Lemma 3.6. *There exists the unique bilinear map $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ satisfying*

$$\nabla_X Y - \nabla_Y X = [X,Y], \qquad X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle \qquad (X,Y,Z \in \mathfrak{X}(M)) \tag{3.12}$$

Definition 3.7. The map ∇ in Lemma 3.6 is called the *Riemannian connection* or the *Levi-Covet connection* of (M,g).

Lemma 3.8. *The Riemannian connection ∇ satisfies*

$$\nabla_{fX} Y = f\nabla_X Y, \qquad \nabla_X(fY) = (Xf)Y + f\nabla_X Y. \tag{3.13}$$

Remark 3.9. A bilinear map $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying (3.13) is called a *linear connection* or an *amine connection*.

Remark 3.10. By Lemmas 3.8 and 3.2, $X \mapsto \nabla_X Y$ determines a one form.

Orthonormal frames. For a sake of simplicity, we assume that g is positive definite, in other words, (M,g) is a Riemannian manifold.

Definition 3.11. Let $U \subset M$ be a domain of M. An n-tuple of vector fields $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ on U is called an *orthonormal frame* on U if $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}$. It is said to be *positive* if M is oriented and $\{\boldsymbol{e}_j\}$ is compatible to the orientation on M.

Remark 3.12. For each $p \in M$, there exists a neighborhood U of p which admits an orthonormal frame on U.

Lemma 3.13. *Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on $U \subset M$. Then there exists a smooth map*

$$\Theta\colon U \longrightarrow \mathrm{O}(n) \qquad \text{such that} \qquad [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n] = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]\Theta. \tag{3.14}$$

Moreover, if $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ determines the common orientation, Θ is valued on $\mathrm{SO}(n)$*.*

The map Θ in Lemma 3.13 is called a *gauge transformation*.

For an orthonormal frame $\{\boldsymbol{e}_j\}$ on U, we denote by $\{\omega^j\}_{j=1,\dots,n}$ the *dual frame* of $\{\boldsymbol{e}_j\}$, that is, $\omega^j \in \wedge^1(U)$ such that

$$\omega^j(\boldsymbol{e}_k) = \delta^j_k = \begin{cases} 1 & (j=k) \\ 0 & (\text{otherwise}). \end{cases}$$

In other words, $\omega^j(X) = \langle \boldsymbol{e}_j, X\rangle$.

Lemma 3.14. *Two orthonormal frames $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ are related as* (3.14)*. Then their duals $\{\omega^j\}$ and $\{\lambda^j\}$ satisfy*

$$\begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} = \Theta \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}.$$

Proof.

$$\begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} (\boldsymbol{e}_1,\dots,\boldsymbol{e}_n) = \begin{pmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{pmatrix} (\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\Theta = \Theta = \Theta \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} (\boldsymbol{e}_1,\dots,\boldsymbol{e}_n). \qquad \square$$

Connection forms.

Definition 3.15. The *connection form* with respect to an orthonormal frame $\{\boldsymbol{e}_j\}$ is a $n\times n$-matrix valued one form Ω on U defined by

$$\Omega = \begin{pmatrix} \omega^1_1 & \omega^1_2 & \dots & \omega^1_n \\ \omega^2_1 & \omega^2_2 & \dots & \omega^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega^n_1 & \omega^n_2 & \dots & \omega^n_n \end{pmatrix}, \qquad \omega^k_j := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle \in \wedge^1(U).$$

By definition, we have $\nabla \boldsymbol{e}_j = \sum_{k=1}^n \omega^k_j \boldsymbol{e}_k$, that is, $\nabla[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n] = [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]\Omega$.

Lemma 3.16. $\omega^k_j = -\omega^j_k$.

Proof. $\omega^k_j = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle = d\langle \boldsymbol{e}_j, \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_j, \nabla \boldsymbol{e}_k\rangle = -\omega^j_k$.

$\square$

Lemma 3.17. $d\omega^i = \sum_{l=1}^n \omega^l \wedge \omega^i_l$.

Proof.

$$\begin{aligned} d\omega^i(\boldsymbol{e}_j,\boldsymbol{e}_k) &= \boldsymbol{e}_j\omega^i(\boldsymbol{e}_k) - \boldsymbol{e}_k\omega^i(\boldsymbol{e}_j) - \omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k]) = -\omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k]) \\ &= -\omega^i(\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j) = -\left\langle \nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j, \boldsymbol{e}_i\right\rangle = -\omega^i_k(\boldsymbol{e}_j) + \omega^i_j(\boldsymbol{e}_k) \\ &= \sum_{l=1}^n \bigl(-\omega^i_l(\boldsymbol{e}_j)\omega^l(\boldsymbol{e}_k) + \omega^i_l(\boldsymbol{e}_k)\omega^l(\boldsymbol{e}_j)\bigr) = \sum_{l=1}^n \omega^l \wedge \omega^i_l(\boldsymbol{e}_j,\boldsymbol{e}_k). \qquad \square \end{aligned}$$

Exercises

3-1 Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as (3.14). Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$.

3-2 Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-1)$ the hyperbolic 2-space (i.e. the hyperbolic plane) of constant curvature -1.

(1) Verify that gives a local coordinate system on $U := H^2(-1) \setminus \{(1,0,0)\}$, and

$$\boldsymbol{e}_1 := (\sinh u, \cos v \cosh u, \sin v \cosh u), \qquad \boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

forms a orthonormal frame on U.

(2) Compute the connection form(s) with respect to the orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.