4 Curvatre forms

4.1 Addendum to the previous section

Proposition 4.1 (The local expression of the Lie bracket)**.** *Let $(U; x^1, \dots, x^n)$ be a coordinate neighborhood of an n-manifold M. Then the Lie bracket of two vector fields*

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

Proof. For a smooth function f on U, it holds that

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} f.$$

Hence $[\partial/\partial x^i, \partial/\partial x^j] = 0$. Then the conclusion follows from bilinearlity of $[X, Y]$ and the formula

$$[fX, Y] = f[X, Y] - (Yf)X, \qquad [X, fY] = f[X, Y] + (Xf)Y$$

for a smooth function f and vector fields X and Y. □

Proposition 4.2 (A local expression of the connection forms)**.** *Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ an orthonormal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as*

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(- \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle + \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i \rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j \rangle \Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

Proof. By the definition of the Levi-Civita connection ∇,

$$\begin{aligned}
\omega_i^j(\boldsymbol{e}_k) &= \langle \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = \boldsymbol{e}_k \langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle - \langle \boldsymbol{e}_i, \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_j \rangle = - \left\langle \boldsymbol{e}_i, \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_k + [\boldsymbol{e}_k, \boldsymbol{e}_j] \right\rangle \\
&= -\boldsymbol{e}_j \langle \boldsymbol{e}_i, \boldsymbol{e}_k \rangle + \left\langle \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_i, \boldsymbol{e}_k \right\rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k] \rangle \\
&= \langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k \rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k] \rangle \\
&= \boldsymbol{e}_i \langle \boldsymbol{e}_j, \boldsymbol{e}_k \rangle - \langle \boldsymbol{e}_j, \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_k \rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k] \rangle \\
&= - \langle \boldsymbol{e}_j, \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_i \rangle - \langle \boldsymbol{e}_j, [\boldsymbol{e}_i, \boldsymbol{e}_k] \rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k] \rangle \\
&= -\omega_i^j(\boldsymbol{e}_k) + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k \rangle - \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i \rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j \rangle .
\end{aligned}$$

□

4.2 Preliminaries

Integrability condition, a review. Let U be a domain of $\mathbb{R}^m$ with coordinate system $(x^1, \dots, x^m)$, and consider a system of differential equations

$$\frac{\partial F}{\partial x^l} = F \Omega_l \qquad (l = 1, \dots, m) \tag{4.1}$$

with initial condition

$$F(\mathrm{P}_0) = F_0 \in \mathrm{M}_n(\mathbb{R}), \qquad \mathrm{P}_0 = (x_0^1, \dots, x_0^m) \in U, \tag{4.2}$$

where F is an unknown map into the space of $n \times n$-real matrices $\mathrm{M}_n(\mathbb{R})$, and the coefficient matrices Ω_l $(l = 1, \dots, m)$ are $\mathrm{M}_n(\mathbb{R})$-valued C^∞-functions.

Lemma 4.3. *If the initial condition F_0 in (4.2) is non-singular, i.e., $F_0 \in \mathrm{GL}(n, \mathbb{R})$*[7]*, F satisfying*

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[7] $\mathrm{GL}(n, \mathbb{R})$ denotes the set of $n \times n$-regular matrices.

(4.1) *is a* $\mathrm{GL}(n,\mathbb{R})$*-valued function, that is,* F *is invertible for each point on* U*.*

Proof. For each $\mathrm{P}\in U$, take a smooth path $\gamma(t):=(x^1(t),\dots,x^m(t))$ $(0\leqq t\leqq 1)$ with $\gamma(0)=\mathrm{P}_0$ and $\gamma(1)=\mathrm{P}$. Then the matrix-valued function $\hat F := F\circ\gamma$ of one variable satisfies the ordinary differential equation

$$\frac{d\hat F}{dt}=\hat F\hat\Omega,\qquad \hat\Omega:=\sum_{l=1}^m \Omega_l\circ\gamma\frac{dx^l}{dt}.$$

Hence $\varphi:=\det\hat F$ satisfies

$$\frac{d\varphi}{dt}=\frac{d}{dt}\det\hat F=\operatorname{tr}\left(\widetilde{\hat F}\frac{d\hat F}{dt}\right)=\operatorname{tr}(\widetilde{\hat F}\hat F\hat\Omega)=\det\hat F\operatorname{tr}\hat\Omega=\varphi\omega$$

where $\widetilde{\hat F}$ denotes the cofactor matrix of $\hat F$ and $\omega:=\operatorname{tr}\hat\Omega$. So

$$\det\hat F(t)=\varphi(t)=\varphi_0\exp\int_0^t\omega(\tau)\,d\tau\qquad(\varphi_0:=\det F_0),$$

proving the lemma. □

As seen in the previous lecturesthe following *integrability condition* holds:

Lemma 4.4. *If a* C^∞*-map* $F\colon U\to\mathrm{GL}(n,\mathbb{R})$ *satisfies* (4.1)*, then it hold on* U *that*

$$\frac{\partial\Omega_l}{\partial x^k}-\frac{\partial\Omega_k}{\partial x^l}+\Omega_k\Omega_l-\Omega_l\Omega_k=O\qquad(1\leqq k<l\leqq m). \tag{4.3}$$

The integrability condition (4.3) guarantees existence of the solution of (4.1) as follows

Theorem 4.5. *Let* $\Omega_l\colon U\to\mathrm{M}_m(\mathbb{R})$ $(l=1,\dots,n)$ *be* C^∞*-functions defined on a simply connected domain* $U\subset\mathbb{R}^n$ *satisfying* (4.3) *Then for each* $\mathrm{P}_0\in U$ *and* $F_0\in\mathrm{M}_m(\mathbb{R})$*, there exists the unique* $m\times m$*-matrix valued function* $F\colon U\to\mathrm{M}_m(\mathbb{R})$ *satisfying* (4.1) *and* (4.2)*. Moreover,*

- *if* $F_0\in\mathrm{GL}(m,\mathbb{R})$*,* $F(\mathrm{P})\in\mathrm{GL}(m,\mathbb{R})$ *holds on* U*,*
- *if* $F_0\in\mathrm{SO}(n)$ *and* Ω_l*'s are skew-symmetric matrices,* $F(\mathrm{P})\in\mathrm{SO}(n)$ *holds on* U*.*

Coordinate-free expressions Let $\Omega_l\colon U\to\mathrm{M}_n(\mathbb{R})$ $(l=1,\dots,m)$ be C^∞-functions defined on a domain $U\subset\mathbb{R}^m$, and define $n\times n$-matrix Ω of 1-forms as

$$\Omega=\begin{pmatrix}\omega_1^1&\omega_2^1&\dots&\omega_n^1\\ \omega_1^2&\omega_2^2&\dots&\omega_n^2\\ \vdots&\vdots&\ddots&\vdots\\ \omega_1^n&\omega_2^n&\dots&\omega_n^n\end{pmatrix}:=\sum_{l=1}^m\Omega_l\,dx^l=\begin{pmatrix}\sum\omega_{l,1}^1\,dx^l&\sum\omega_{l,2}^1\,dx^l&\dots&\sum\omega_{l,n}^1\,dx^l\\ \sum\omega_{l,1}^2\,dx^l&\sum\omega_{l,2}^2\,dx^l&\dots&\sum\omega_{l,n}^2\,dx^l\\ \vdots&\vdots&\ddots&\vdots\\ \sum\omega_{l,1}^n\,dx^l&\sum\omega_{l,2}^n\,dx^l&\dots&\sum\omega_{l,n}^n\,dx^l\end{pmatrix}, \tag{4.4}$$

where $\Omega_l=(\omega_{l,j}^i)$. Then Ω is considered as a $\mathrm{M}_n(\mathbb{R})$-valued 1-form, and (4.1) is restated as

$$dF=F\Omega. \tag{4.5}$$

Lemma 4.6. *Under the situation above, the integrability condition* (4.3) *is equivalent to*

$$d\Omega+\Omega\wedge\Omega=O,\qquad \textit{where}\quad \Omega\wedge\Omega=\left(\sum_{k=1}^n\omega_k^i\wedge\omega_j^k\right)_{i,j=1,\dots,n}. \tag{4.6}$$

Proof. Assume F be a solution of (4.5) with $F \in \mathrm{GL}(n,\mathbb{R})$. Then

$$O = ddF = d(F\Omega) = dF \wedge \Omega + F\,d\Omega = F(\Omega \wedge \Omega + d\Omega). \qquad \square$$

Thus, by using differential forms, we can state the system of partial differential equations (4.1) and its integrability condition (4.3) in coordinate-free form. The proof of Theorem 4.5 works not only simply connected domain $U \subset \mathbb{R}^m$ but also simply connected m-manifold, and thus, we have

Theorem 4.7. *Let Ω be an $\mathrm{M}_n(\mathbb{R})$-valued 1-form on a simply connected m-manifold M satisfying (4.6). Then for each $\mathrm{P}_0 \in M$ and $F_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n\times n$-matrix valued function $F\colon M \to \mathrm{M}_n(\mathbb{R})$ satisfying (4.5) with $F(\mathrm{P}) = F_0$. Moreover,*

- *if $F_0 \in \mathrm{GL}(n,\mathbb{R})$, $F(\mathrm{P}) \in \mathrm{GL}(n,\mathbb{R})$ holds on M,*
- *if $F_0 \in \mathrm{SO}(m)$ and Ω is skew-symmetric, $F(\mathrm{P}) \in \mathrm{SO}(m)$ holds on M.*

When $n = 1$, that is, Ω is a usual 1-form, $\Omega \wedge \Omega$ always vanishes, and the integrability condition (4.6) is simply $d\Omega = 0$. Then we have the following Poncaré's lemma[8].

Theorem 4.8 (Poincaré's lemma)**.** *If a differential 1-form ω defined on a simply connected and connected m-manifold M is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.*

Proof. Since ω is closed, there exists a function F on M satisfying $dF = F\omega$ with initial condition $F(\mathrm{P}_0) = 1$. By Lemma 4.3, F does not vanish on M, that is, $F > 0$. Hence $f := \log F$ is a smooth function on M satisfying $df = dF/F = F\omega/F = \omega$. Take another function g on M satisfying $dg = \omega$, $d(f - g) = 0$ holds. Then connectedness of M infers that $f - g$ is constant. $\square$

4.3 Curvature form

Let U be a domain of n-dimensional Riemannian manifold (M, g). We let Ω be the connection form with respect to an orthonormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on U, as defined in Definition 3.15.

Definition 4.9. We define a skew-symmetric matrix-valued 2-form by $K := d\Omega + \Omega \wedge \Omega$ and call the *curvature form* with respect to the frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$.

Take an orthonormal frame $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ on U and take a gauge transformation $\Theta\colon U \to \mathrm{O}(n)$:

$$[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Denoting the connection form and the curvature form with respect to $[\boldsymbol{v}_j]$ by $\widetilde{\Omega}$ and $\widetilde{K}$. Then

Proposition 4.10. (1) $\Omega = \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta$, (2) $K = \Theta^{-1}\widetilde{K}\Theta$.

Proof. Since

$$\begin{aligned}[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega &= \nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = \nabla([\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta) = \nabla[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta + [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]d\Theta \\ &= [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\widetilde{\Omega}\Theta + [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]d\Theta = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Theta^{-1}(\widetilde{\Omega}\Theta + d\Theta),\end{aligned}$$

the first assertion is obtained. Next, noticing $d(\widetilde{\Omega}\Theta) = (d\widetilde{\Omega})\Theta - \widetilde{\Omega} \wedge d\Theta$, $\widetilde{\Omega}\Theta^{-1} \wedge \Theta\widetilde{\Omega} = \widetilde{\Omega} \wedge \widetilde{\Omega}$, and so on, we have

$$\begin{aligned}d\Omega + \Omega \wedge \Omega &= d(\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) + (\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) \wedge (\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) \\ &= -\Theta^{-1}d\Theta\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\widetilde{\Omega}\Theta - \Theta^{-1}\widetilde{\Omega} \wedge d\Theta - \Theta^{-1}d\Theta\Theta^{-1} \wedge d\Theta \\ &\quad + \Theta^{-1}\widetilde{\Omega}\Theta \wedge \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}\widetilde{\Omega}\Theta \wedge \Theta^{-1}d\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}d\Theta \\ &= \Theta^{-1}(d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega})\Theta,\end{aligned}$$

proving (2). $\square$

[8] Theorem 2.6 in Advanced Topics in Geometry E (MTH.B501).

The goal of this section is to prove the following

Theorem 4.11. *Let U be a domain of a Riemannian n-manifold (M, g) and K the curvature form with respect to an orthonormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on U. For a point $\mathrm{P} \in U$, there exists a local coordinate system $(x^1, \dots, x^n)$ around P such that $[\partial/\partial x^1, \dots, \partial/\partial x^n]$ is an orthonormal frame if and only if K vanishes on a neighborhood of P.*

Remark 4.12. By (2) of Proposition 4.10, the condition $K = 0$ does not depend on choice of orthonormal frames. A Riemannian manifold (M, g) said to be *flat* if $K = 0$ holds on M.

Proof of Theorem 4.11. First, we shall show the "only if" part: Let $(x^1, \dots, x^n)$ be a coordinate system such that $[\boldsymbol{e}_j := \partial/\partial x^j]$ is an orthonormal frame. Since

$$[\boldsymbol{e}_j, \boldsymbol{e}_k] = \left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] = \mathbf{0},$$

Proposition 4.2 yields that all components of the connection forms ω_i^j vanish. Hene we have $K = 0$.

Conversely, assume $K = 0$ for an orthonormal frame $[\boldsymbol{e}_j]$. Since the connection form $\varOmega$ satisfies $d\varOmega + \varOmega \wedge \varOmega = O$, there exists a matrix-valued function $\varTheta \colon V \to \mathrm{SO}(n)$ satisfying $d\varTheta = \varTheta\varOmega$, $\varTheta(\mathrm{P}) = \mathrm{id}$ on a sufficiently small neighborhood V of P, because of Theorem 4.5. Take a new orthonormal frame $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] := [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\varTheta^{-1}$. Then by (1) of Proposition 4.10, the connection form $\widetilde{\varOmega} = (\tilde{\omega}_i^j)$ with respect to $[\boldsymbol{v}_j]$ vanishes identically. So by Lemma 3.17, $d\omega^i = 0$ holds for $i = 1, \dots, n$. Hence by the Poincaré Lemma (Theorem 4.8), there exists a smooth functions on a neighborhood V of P. Such $(x^1, \dots, x^n)$ is a desired coordinate system if V is sufficiently small. □

Exercises

4-1 Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \text{on} \qquad U := \{(r, \theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ and φ is a positive smooth function defined on $(0, r_0)$ with

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \varphi'(r) = 1.$$

Find a function φ such that (U, g) is flat. (Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ is an orthonormal frame.)

4-2 Compute the curvature form of $H^2(-1)$ with respect to an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ as in Exercise 3-2.