# 5 The Sectional Curvature

## *5.1 Preliminaries*

**Exterior products of tangent vectors.** Let $V$ be an $n$-dimensional vector space ($1 \leqq n < \infty$) and denote by $V^*$ its dual. Then $(V^*)^*$ can be naturally identified with $V$ itself. In fact,

$$I : V \ni \boldsymbol{v} \longmapsto I_{\boldsymbol{v}} \in (V^*)^* := \{A\colon V^* \to \mathbb{R}; \text{linear}\}, \qquad I_{\boldsymbol{v}}(\alpha) := \alpha(\boldsymbol{v})$$

is a linear map with trivial kernel. Then $I$ is an isomorphism because $\dim(V^*)^* = \dim V$.

We denote by $\wedge^2 V := \wedge^2(V^*)^*$ the set of skew-symmetric bilinear forms on $V^*$. For vectors $\boldsymbol{v}$, $\boldsymbol{w} \in V$, the *exterior product* of them is an element of $\wedge^2 V$ defined as

$$(\boldsymbol{v} \wedge \boldsymbol{w})(\alpha, \beta) := \alpha(\boldsymbol{v})\beta(\boldsymbol{w}) - \alpha(\boldsymbol{w})\beta(\boldsymbol{v}) \qquad (\alpha, \beta \in V^*).$$

For a basis $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on $V$,

$$\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j ; 1 \leqq i < j \leqq n\} \tag{5.1}$$

is a basis of $\wedge^2 V$. In particular $\dim \wedge^2 V = \frac{1}{2}n(n-1)$. When $V$ is a vector space endowed with an inner product $\langle\ ,\ \rangle$ and $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ is an orthonormal basis, there exists the unique inner product, which is also denoted by $\langle\ ,\ \rangle$, of $\wedge^2 V$ such that (5.1) is an orthonormal basis. This definition of the inner product does not depend on choice of orthonormal bases of $V$. In fact, take another orthonormal basis $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ related with $[\boldsymbol{e}_j]$ by

$$[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta \qquad \Theta = (\theta_i^j) \in \mathrm{O}(n).$$

Since $\Theta^T = \Theta^{-1}$, $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Theta^T$ holds. Hence

$$\boldsymbol{v}_s \wedge \boldsymbol{v}_t = \left(\sum_i \theta_s^i \boldsymbol{e}_i\right) \wedge \left(\sum_j \theta_t^j \boldsymbol{e}_j\right) = \sum_{i,j} \theta_i^s \theta_j^t (\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \sum_{i<j} \left(\theta_i^s \theta_j^t - \theta_j^s \theta_i^t\right)(\boldsymbol{e}_i \wedge \boldsymbol{e}_j),$$

and so

$$\begin{aligned}
\langle \boldsymbol{v}_s \wedge \boldsymbol{v}_t, \boldsymbol{v}_u \wedge \boldsymbol{v}_v \rangle &= \sum_{i<j,k<l} (\theta_i^s \theta_j^t - \theta_j^s \theta_i^t)(\theta_k^u \theta_l^v - \theta_l^u \theta_k^v) \langle \boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_k \wedge \boldsymbol{e}_l \rangle \\
&= \sum_{i<j,k<l} (\theta_i^s \theta_j^t - \theta_j^s \theta_i^t)(\theta_k^u \theta_l^v - \theta_l^u \theta_k^v)\delta_{ik}\delta_{jl} = \sum_{i<j} (\theta_i^s \theta_j^t - \theta_j^s \theta_i^t)(\theta_i^u \theta_j^v - \theta_j^u \theta_i^v) \\
&= \sum_{i<j} (\theta_i^s \theta_j^t \theta_i^u \theta_j^v - \theta_j^s \theta_i^t \theta_i^u \theta_j^v - \theta_i^s \theta_j^t \theta_j^u \theta_i^v + \theta_j^s \theta_i^t \theta_j^u \theta_i^v) \\
&= \sum_{i<j} \theta_i^s \theta_j^t \theta_i^u \theta_j^v + \sum_{i<j} \theta_j^s \theta_i^t \theta_i^u \theta_j^v - \sum_{i>j} \theta_j^s \theta_i^t \theta_i^u \theta_j^v + \sum_{i>j} \theta_i^s \theta_j^t \theta_i^u \theta_j^v \\
&= \sum_{i\neq j} \theta_i^s \theta_j^t \theta_i^u \theta_j^v - \sum_{i\neq j} \theta_j^s \theta_i^t \theta_i^u \theta_j^v \\
&= \sum_{i,j} (\theta_i^s \theta_j^t \theta_i^u \theta_j^v - \theta_j^s \theta_i^t \theta_i^u \theta_j^v) - \sum_i (\theta_i^s \theta_i^t \theta_i^u \theta_i^v - \theta_i^s \theta_i^t \theta_i^u \theta_i^v) \\
&= \delta^{su}\delta^{tv} - \delta^{tu}\delta^{sv}
\end{aligned}$$

because $\sum_i \theta_i^s \theta_i^t = \delta^{st}$. So, if $s < t$ and $u < v$, the second term of the right-hand side vanishes. That is, $\{\boldsymbol{v}_s \wedge \boldsymbol{v}_t\,;\, s < t\}$ is an orthonormal basis as well as $\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\,;\, i < j\}$ is.

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**Symmetric bilinear forms.** Let $V$ be a real vector space. A bilinear map $q\colon V\times V\to\mathbb{R}$ is said to be *symmetric* if $q(\boldsymbol{v},\boldsymbol{w})=q(\boldsymbol{w},\boldsymbol{v})$ for all $\boldsymbol{v}$, $\boldsymbol{w}\in V$.

**Lemma 5.1.** *Two symmetric bilinear forms $q$ and $q'$ coincide with each other if and only if $q(\boldsymbol{v},\boldsymbol{v})=q'(\boldsymbol{v},\boldsymbol{v})$ hold for all $\boldsymbol{v}\in V$.*

*Proof.* By symmetricity, $q(\boldsymbol{v},\boldsymbol{w})=\frac{1}{2}(q(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w})-q(\boldsymbol{v},\boldsymbol{v})-q(\boldsymbol{w},\boldsymbol{w}))$ holds. □

### 5.2 Sectional Curvature

Let $U$ be a domain on a Riemannian $n$-manifold $(M,g)$, and $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ an orthonormal frame on $U$. Denote by $(\omega^j)_{j=1,\dots,n}$, $\varOmega=(\omega^j_i)_{i,j=1,\dots,n}$ and $K=(\kappa^j_i)_{i=1,\dots,n}:=d\varOmega+\varOmega\wedge\varOmega$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$. Then Lemma 3.17 and Definition 4.9, we have

$$d\omega^j=\sum_l\omega^l\wedge\omega^j_l,\qquad \kappa^j_i=d\omega^j_i+\sum_l\omega^j_l\wedge\omega^l_i. \tag{5.2}$$

Since $\varOmega$ is a one form valued in the skew-symmetric matrices, so is $K$:

$$\omega^j_i=-\omega^i_j,\qquad \kappa^j_i=-\kappa^i_j. \tag{5.3}$$

**Proposition 5.2** (The first Bianchi identity)**.** $\kappa^i_j(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa^i_k(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa^i_l(\boldsymbol{e}_j,\boldsymbol{e}_k)=0$.

*Proof.* By (5.2) and (3.11),

$$\begin{aligned}
0=dd\omega^i&=d\left(\sum_s\omega^s\wedge\omega^i_s\right)=\sum_s\left(d\omega^s\wedge\omega^i_s-\omega^s\wedge\omega^i_s\right)\\
&=\sum_s\left(\sum_m(\omega^m\wedge\omega^s_m)\wedge\omega^i_s-\omega^s\wedge\left(\kappa^i_s-\sum_m\omega^i_m\wedge d\omega^m_s\right)\right)\\
&=\sum_{s,m}\omega^m\wedge\omega^s_m\wedge\omega^i_s+\sum_{s,m}\omega^s\wedge\omega^i_m\wedge\omega^m_s-\sum_s\omega^s\wedge\kappa^i_s\\
&=\sum_{s,m}\omega^m\wedge(\omega^s_m\wedge\omega^i_s+\omega^i_s\wedge\omega^s_m)-\sum_s\omega^s\wedge\kappa^i_s=-\sum_s\omega^s\wedge\kappa^i_s.
\end{aligned}$$

Hence

$$\begin{aligned}
0&=\sum_s(\omega^s\wedge\kappa^i_s)(\boldsymbol{e}_j,\boldsymbol{e}_k,\boldsymbol{e}_l)=\sum_s\big(\omega^s(\boldsymbol{e}_j)\kappa^i_s(\boldsymbol{e}_k,\boldsymbol{e}_l)+\omega^s(\boldsymbol{e}_k)\kappa^i_s(\boldsymbol{e}_l,\boldsymbol{e}_j)+\omega^s(\boldsymbol{e}_l)\kappa^i_s(\boldsymbol{e}_j,\boldsymbol{e}_k)\big)\\
&=\sum_s\big(\delta^s_j\kappa^i_s(\boldsymbol{e}_k,\boldsymbol{e}_l)+\delta^s_k\kappa^i_s(\boldsymbol{e}_l,\boldsymbol{e}_j)+\delta^s_l\kappa^i_s(\boldsymbol{e}_j,\boldsymbol{e}_k)\big)\\
&=\kappa^i_j(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa^i_k(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa^i_l(\boldsymbol{e}_j,\boldsymbol{e}_k),
\end{aligned}$$

proving the assertion. □

**Corollary 5.3.** $\kappa^i_j(\boldsymbol{e}_k,\boldsymbol{e}_l)=\kappa^k_l(\boldsymbol{e}_i,\boldsymbol{e}_j)$.

*Proof.* By Proposition 5.2,

$$\begin{aligned}
\kappa^i_j(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa^i_k(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa^i_l(\boldsymbol{e}_j,\boldsymbol{e}_k)=0\\
\kappa^j_k(\boldsymbol{e}_i,\boldsymbol{e}_l)+\kappa^j_i(\boldsymbol{e}_l,\boldsymbol{e}_k)+\kappa^j_l(\boldsymbol{e}_k,\boldsymbol{e}_i)=0\\
\kappa^k_i(\boldsymbol{e}_j,\boldsymbol{e}_l)+\kappa^k_j(\boldsymbol{e}_l,\boldsymbol{e}_i)+\kappa^k_l(\boldsymbol{e}_i,\boldsymbol{e}_j)=0.
\end{aligned}$$

Summing up these and noticing $\kappa^j_i=-\kappa^i_j$, we have the conclusion. □

**A quadratic form induced from the curvature form.** We fix a point $p \in U$. Under the notation above, we can define a bilinear map

$$(5.4) \qquad \boldsymbol{K}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{i<j, k<l} \kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l)\xi^{kl}\eta^{ij}, \qquad \boldsymbol{\xi} = \sum_{k<l} \xi^{kl}\boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j} \eta^{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

on $\wedge^2 T_pM$, where $\boldsymbol{e}_j$, $\kappa_i^j$...are considered tangent vectors, 2-forms at the fixed point $p$. In fact, one can show that the definition (5.4) is independent of choice of orthonormal frames. As a immediate conclusion of Corollary 5.3, we have

**Lemma 5.4.** *$\boldsymbol{K}$ is symmetric.*

Hence, taking Lemma 5.1 into an account, we define the sectional curvature as follows:

**Definition 5.5.** Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in $T_pM$. The *sectional curvature* of $(M, g)$ with respect to the plane $\Pi_p$ is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of $\Pi_p$

*Remark* 5.6. For (not necessarily orthonormal) basis $\{\boldsymbol{x}, \boldsymbol{y}\}$ of $\Pi_p$, the sectional curvature is expressed as

$$K(\Pi_p) = \frac{\boldsymbol{K}(\boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \wedge \boldsymbol{y})}{\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{x} \wedge \boldsymbol{y} \rangle},$$

where $\langle\ ,\ \rangle$ of the right-hand side is the inner product of $\wedge^2 T_pM$ induced from the Riemannian metric.

*Remark* 5.7. The sectional curvature is a scalar corresponding to a 2-plane in the tangent space $T_pM$. Hence it can be considered as a function of 2-Grassmanian bundle induced from the tangent bundle $TM$.

### *5.3 Curvature Tensor*

Let $(M, g)$ be a Riemannian manifold and $\nabla$ the Levi-Civita connection. Define a trilinear map

$$(5.5) \quad R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y, Z) \mapsto R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \in \mathfrak{X}(M).$$

By the properties Lemma 3.6 of the connection and the property (3.4) of the Lie bracket, the following Lemma is obvious.

**Lemma 5.8.** *For any function $f \in \mathcal{F}(M)$ and vector fields $X$, $Y$, $Z \in \mathfrak{X}(M)$,*

$$R(fX, Y)Z = R(X, fY)Z = R(X, Y)(fZ) = fR(X, Y)Z$$

*holds.*

**Corollary 5.9.** *Assume the vector fields $X$, $Y$, $Z$ and $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \mathfrak{X}(M)$ satisfy $X_p = \widetilde{X}_p$, $Y_p = \widetilde{Y}_p$ and $Z_p = \widetilde{Z}_p$ for a point $p \in M$. Then*

$$\bigl(R(X, Y)Z\bigr)_p = \bigl(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}\bigr)_p.$$

*In other words, $R$ in (5.5) induces a trilinear map*

$$R_p\colon T_pM \times T_pM \times T_pM \to T_pM.$$

**Definition 5.10.** A trilinear map $R(X,Y)Z$ is called the *curvature tensor* of $(M,g)$. In addition, a quadrilinear map

$$R(X,Y,Z,T) = \langle R(X,Y)Z,T\rangle : \mathfrak{X}(M)^4 \to \mathcal{F}(M)$$

is also called the *curvature tensor*. In fact, $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$, that is $R$ is $(0,4)$-tensor field, because $R$ induces a quadrilinear map

$$R\colon (T_pM)^4 \to \mathbb{R}$$

for each $p \in M$.

**Lemma 5.11.** *Let $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\}$ be an orthonormal frame on a domain $U \subset M$, and $K = (\kappa_i^j)$ the curvature form with respect to the frame. Then it holds that*

$$\kappa_i^j(X,Y) = R(X,Y,\boldsymbol{e}_i,\boldsymbol{e}_j)$$

*for each $(i,j)$.*

So by (5.3), Proposition 5.2, Corollary 5.3 yield

**Proposition 5.12.**

- $R(X,Y,Z,T) = -R(Y,X,Z,T) = -R(X,Y,T,Z)$,
- $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$,
- $R(X,Y,Z,T) = R(Z,T,X,Y)$.

Moreover, the sectional curvature $K(\Pi_p)$ in Definition 5.5 is computed by

$$K(\Pi_p) = \frac{R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x})}{\langle \boldsymbol{x},\boldsymbol{x}\rangle\langle \boldsymbol{y},\boldsymbol{y}\rangle - \langle \boldsymbol{x},\boldsymbol{y}\rangle^2}. \tag{5.6}$$

***Exercises***

**5-1** Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\,d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0<r<r_0, -\pi<\theta<\pi\},$$

where $r_0 \in (0,+\infty]$ and $\varphi$ is a positive smooth function defined on $(0,r_0)$ with

$$\lim_{r\to+0}\varphi(r) = 0, \qquad \lim_{r\to+0}\frac{\varphi(r)}{r} = 1.$$

Classify the function $\varphi$ so that $g$ is of constant sectional curvature.

**5-2** Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces a smooth map

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of $C^\infty$-functions. Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis $(\omega^j)$ to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^{n} \boldsymbol{e}_j\omega^j$$

holds on $U$. Here, we regard that $d\boldsymbol{x}$ is an $(n+1)$-tuple of differential forms and $\boldsymbol{e}_j$ is an $\mathbb{R}^{n+1}$-valued function for each $j$.