6 Space forms

6.1 Constant sectional curvature

Let (M, g) be a Riemannian n-manifold, and let

$$\begin{aligned}&\mathrm{Gr}_2(TM) := \cup_p \mathrm{Gr}_2(T_pM),\\ &\qquad \mathrm{Gr}_2(T_pM) := \text{2-Grassmannian of } T_pM = \{\Pi_p \subset T_pM\,;\, \text{2-dimensional subspace}\}.\end{aligned}$$

The sectional curvature defined in Definition 5.5 is a map $K\colon \mathrm{Gr}_2(TM) \to \mathbb{R}$ such that

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v}\wedge\boldsymbol{w}, \boldsymbol{v}\wedge\boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is the orthonormal basis of Π_p.

Fix a point p, and take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ defined on a neighborhood U of p. Denote by (ω^j), $\Omega = (\omega_i^j)$ and $K = (\kappa_i^j)$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$, respectively.

Theorem 6.1. *Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as*

$$\kappa_j^i = k\omega^i \wedge \omega^j.$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

Proof. By the assumption, $k = K(\mathrm{Span}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}) = \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_j, \boldsymbol{e}_i\wedge\boldsymbol{e}_j) = \kappa_j^i(\boldsymbol{e}_i, \boldsymbol{e}_j)$. Let

$$\boldsymbol{v} := \cos\theta\boldsymbol{e}_i + \sin\theta\boldsymbol{e}_j, \qquad \boldsymbol{w} := \cos\varphi\boldsymbol{e}_l + \sin\varphi\boldsymbol{e}_m$$

where $\{i, j\} \neq \{l, m\}$, and set $\Pi_{\theta,\varphi} := \mathrm{Span}\{\boldsymbol{v}, \boldsymbol{w}\} \subset T_pM$. Then by biliniearity of the $\wedge$-product on T_pM, it holds that

$$\boldsymbol{v}\wedge\boldsymbol{w} = \cos\theta\cos\varphi\boldsymbol{e}_i\wedge\boldsymbol{e}_l + \cos\theta\sin\varphi\boldsymbol{e}_i\wedge\boldsymbol{e}_m + \sin\theta\cos\varphi\boldsymbol{e}_j\wedge\boldsymbol{e}_l + \sin\theta\sin\varphi\boldsymbol{e}_j\wedge\boldsymbol{e}_m.$$

Since $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of $\Pi_{\theta,\varphi}$, biliniearity and symmetricity of $\boldsymbol{K}$ implies

$$\begin{aligned}k =&K(\Pi_{\theta,\varphi}) = \boldsymbol{K}(\boldsymbol{v}\wedge\boldsymbol{w}, \boldsymbol{v}\wedge\boldsymbol{w})\\
=&\cos^2\theta\cos^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_i\wedge\boldsymbol{e}_l) + \cos^2\theta\sin^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_i\wedge\boldsymbol{e}_m)\\
&+ \sin^2\theta\cos^2\varphi\boldsymbol{K}(\boldsymbol{e}_j\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_l) + \sin^2\theta\sin^2\varphi\boldsymbol{K}(\boldsymbol{e}_j\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_m)\\
&+ 2\cos^2\theta\cos\varphi\sin\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_i\wedge\boldsymbol{e}_m) + 2\cos\theta\sin\theta\cos^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_l)\\
&+ 2\cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_l))\\
&+ 2\cos\theta\sin\theta\sin^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) + 2\sin^2\theta\cos\varphi\sin\varphi\boldsymbol{K}(\boldsymbol{e}_j\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m)\\
=&k + 2\big(\cos^2\theta\cos\varphi\sin\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_i\wedge\boldsymbol{e}_m) + \cos\theta\sin\theta\cos^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_l)\\
&+ \cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_l))\\
&+ \cos\theta\sin\theta\sin^2\varphi\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) + \sin^2\theta\cos\varphi\sin\varphi\boldsymbol{K}(\boldsymbol{e}_j\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m)\big).\end{aligned} \tag{6.1}$$

So, by letting $\theta = 0$, we have

$$\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_i\wedge\boldsymbol{e}_m) = 0. \tag{6.2}$$

Similarly, letting $\theta = \pi/2$, $\varphi = 0$ and $\varphi = \pi/2$, we have $\boldsymbol{K}(\boldsymbol{e}_j\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) = \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_l) = \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) = 0$. Hence the equality (6.1) implies

$$\boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_l, \boldsymbol{e}_j\wedge\boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i\wedge\boldsymbol{e}_m, \boldsymbol{e}_j\wedge\boldsymbol{e}_l) = 0.$$

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By definition (5.4), this is equivalent to

$$\kappa_j^m(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_j^l(\boldsymbol{e}_i, \boldsymbol{e}_m) = -(\kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_l^j(\boldsymbol{e}_i, \boldsymbol{e}_m)).$$

Then by Proposition 5.2, we have

$$0 = \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_l^j(\boldsymbol{e}_i, \boldsymbol{e}_m) = \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) - \kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) - \kappa_m^j(\boldsymbol{e}_l, \boldsymbol{e}_i) = 2\kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) - \kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l).$$

Exchanging the roles of i and m, it holds that $2\kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) - \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) = 0$. So we have

$$\kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) = 0 \qquad (\text{if } \{i,j\} \neq \{m,l\}). \tag{6.3}$$

On the other hand, (6.2) means that $\kappa_i^j(\boldsymbol{e}_i, \boldsymbol{e}_l) = \kappa_i^j(\boldsymbol{e}_j, \boldsymbol{e}_l) = 0$ when $l \neq i, j$. Summing up, we have

$$\kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \begin{cases} k & (i,j) = (k,l) \\ 0 & \text{otherwise,} \end{cases}$$

proving the theorem. □

We now consider the case that the assumption of Theorem 6.1 holds for each $p \in M$.

Theorem 6.2. *Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p) = k(p)$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \to k(p) \in \mathbb{R}$ is constant provided that M is connected.*

Proof. By taking the exterior derivative of $\kappa_i^j = d\omega_i^j + \sum_s \omega_s^j \wedge \omega_i^s$, it holds that

$$\begin{aligned} d\kappa_i^j &= d(d\omega_i^j) + \sum_s \omega_s^j \wedge d\omega_i^s - \sum_s d\omega_s^j \wedge \omega_i^s \\ &= \sum_s \left(\kappa_s^j - \sum_t \omega_t^j \wedge \omega_s^t\right) \wedge \omega_i^s - \sum_s \omega_s^j \wedge \left(\kappa_i^s - \sum_t \omega_t^s \wedge \omega_i^t\right), \end{aligned}$$

and hence we have the identity

$$d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s\right), \tag{6.4}$$

which is known as the *second Bianchi identity*. By our assumption, Theorem 6.1 implies that $\kappa_i^j = k\omega^i \wedge \omega^j$. Then by Lemma 3.17,

$$\begin{aligned} d\kappa_i^j &= d(k\omega^i) \wedge \omega^j - k\omega^i \wedge d\omega^j = dk \wedge \omega^i \wedge \omega^j + kd\omega^i \wedge \omega^j - k\omega^i \wedge d\omega^j \\ &= dk \wedge \omega^i \wedge \omega^j + \sum_s k\omega^s \wedge \omega_s^i \wedge \omega^j - \sum_s k\omega^i \wedge \omega^s \wedge \omega_s^j = dk \wedge \omega^i \wedge \omega^j + d\kappa_i^j \end{aligned}$$

holds for each i and j. Thus, $dk \wedge \omega^i \wedge \omega^j = 0$ for all i and j, which implies $dk = 0$. This equality is independent of choice of orthonormal frames. Since M is connected, k is constant. □

6.2 Space forms

Let (M, g) be a Riemannian n-manifold. A path $\gamma\colon [0, +\infty) \to M$ is said to be a *divergence path* if for any compact subset $K \in M$, there exists $t_0 \in (0, +\infty)$ such that $\gamma([t_0, +\infty)) \subset M \setminus K$. If any divergent path has infinite length, (M, g) is said to be complete.[9] In particular, a compact Riemannian manifold without boundary is automatically complete.

[9] Usually, completeness is defined in terms of geodesics: A Riemannian manifold (M, g) is complete if any geodesics are defined on entire $\mathbb{R}$. The definition here is one of the equivalent conditions of completeness, expressed in the *Hopf-Rinow theorem. cf. MTH.B505.*

Definition 6.3. An n-dimensional *space form* is a complete Riemannian n-manifold of constant sectional curvature.

Example 6.4. The Euclidean n-space is a space form of constant sectional curvature 0. In fact, let $(x^1, \dots, x^n)$ be the canonical Cartesian coordinate system and set $\boldsymbol{e}_j = \partial/\partial x^j$. Then $[\boldsymbol{e}_j]$ is an orthornormal frame defined on the entire $\mathbb{R}^n$, and Propositions 4.1 and 4.2 implies that the connection form $\omega^i_j = 0$. Hence the curvature forms vanish, and then the sectional curvature is identically zero.

So it is sufficient to show completeness. Let $\gamma\colon [0, +\infty) \to \mathbb{R}^n$ be a divergent path. Then for each $r > 0$, there exists $t_0 > 0$ such that $|\gamma(t)| > r$ holds on $[t_0, +\infty)$, equivalently, $|\gamma(t)| \to +\infty$ as $t \to +\infty$. So the length L of the curve γ is

$$L = \lim_{t\to+\infty} \int_0^t |\dot\gamma(\tau)|\, d\tau \geqq \lim_{t\to+\infty} \left| \int_0^t \dot\gamma(\tau)\, d\tau \right| = \lim_{t\to+\infty} |\gamma(t) - \gamma(0)| \geqq \lim_{t\to+\infty} |\gamma(t)| - |\gamma(0)| = +\infty.$$

Here, we used the triangle inequality of integrals for vector-valued functions[10].

6.3 The Hyperbolic spaces

Let $H^n(-c^2)$ be the hyperbolic n-space defined, where c is a non-zero constant:

$$H^n(-c^2) := \left\{ \boldsymbol{x} = (x^0, \dots, x^n) \in \mathbb{R}^{n+1}_1 \;\middle|\; \langle \boldsymbol{x}, \boldsymbol{x} \rangle_L = -\frac{1}{c^2}, c x_0 > 0 \right\},$$

where $(\mathbb{R}^{n+1}_1, \langle\ ,\ \rangle_L)$ be the Lorentz-Minkowski $(n+1)$-space. The tangent space $T_{\boldsymbol{x}} H^n(-c^2)$ is the orthogonal complement $\boldsymbol{x}^\perp$ of $\boldsymbol{x}$, and the restriction g_H of the inner product $\langle\ ,\ \rangle_L$ to $T_{\boldsymbol{x}} H^n(-c^2)$ is positive definite. Thus, $(H^n(-c^2), g_H)$ is a Riemannian manifold, called the *hyperbolic n-space.*

Theorem 6.5. *The hyperbolic space $(H^n(-c^2), g_H)$ is of constant sectional curvature $-c^2$.*

Proof. Notice that $H^n(-c^2)$ can be expressed as a graph $x^0 = \frac{1}{c}\sqrt{1 + c^2\left((x^1)^2 + \dots + (x^n)^2\right)}$ defined on the $(x^1, \dots, x^n)$-hyperplane, that is, it is covered by single chart. Then there exists a orthonormal frame field $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ defined on entire $H^n(-c^2)$. Denote by (ω^i), $\Omega = (\omega^j_i)$ and $K = (\kappa^j_i)$ the dual frame, the connection form and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively.

Regarding $T_{\boldsymbol{x}} H^n(-c^2)$ as a linear subspace in $\mathbb{R}^{n+1}_1$, we can consider $\boldsymbol{e}_j$ as a vector-valued function. In addition the position vector $\boldsymbol{x} \in H^n(-c^2)$ can be also regarded as a vector-valued function. Since $T_{\boldsymbol{x}} H^n(-c^2) = \boldsymbol{x}^\perp$,

$$\mathcal{F} := (\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n)\colon H^n(-c^2) \to \mathrm{M}_{n+1}(\mathbb{R}) \qquad \boldsymbol{e}_0 = c\boldsymbol{x} \tag{6.5}$$

gives a pseudo orthornormal frame along $H^n(-c^2)$, that is, $\mathcal{F}^T Y \mathcal{F} = Y$ $(Y := \mathrm{diag}(-1, 1, \dots, 1))$ holds.

As seen in Exercise 5-2, it holds that

$$d\boldsymbol{e}_0 = c\, d\boldsymbol{x} = c \sum_{j=1}^n \omega^j \boldsymbol{e}_j. \tag{6.6}$$

On the other hand, for each $j = 1, \dots, n$, decompose the vector-valued one form $d\boldsymbol{e}_j$ as

$$d\boldsymbol{e}_j = h_j \boldsymbol{e}_0 + \sum_s \alpha^s_j \boldsymbol{e}_s,$$

[10] See, for example, Theorem A.1.4 in [UY17] for $n = 2$. The idea of the proof works for general n.

where h_j and α_j^s are one forms on $H^n(-c^2)$. Here,

$$h_j = -\langle d\boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L = -d\langle \boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L + \langle \boldsymbol{e}_j, d\boldsymbol{e}_0\rangle_L = c\omega^j,$$

and

$$\alpha_j^s = \langle d\boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L = d\langle \boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L - \langle \boldsymbol{e}_j, d\boldsymbol{e}_s\rangle_L = -\alpha_s^j.$$

Differentiating (6.6), it holds that

$$0 = \frac{1}{c}\, dd\boldsymbol{e}_0 = \sum_j (d\omega^j \boldsymbol{e}_j - \omega^j \wedge d\boldsymbol{e}_j) = \sum_{j,s} \omega^s \wedge \omega_s^j \boldsymbol{e}_j - \sum_{j,s} \omega^j \wedge \alpha_j^s \boldsymbol{e}_s = \sum_j \sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\boldsymbol{e}_j$$

because $\omega^j \wedge \omega^j = 0$. Thus, we have $\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j) = 0$, and then

$$\begin{aligned}
0 &= \left(\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\right)(\boldsymbol{e}_l, \boldsymbol{e}_m) = (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)) - (\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)),\\
0 &= (\omega_j^m(\boldsymbol{e}_l) - \alpha_j^m(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) = -(\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)),\\
0 &= (\omega_m^l(\boldsymbol{e}_j) - \alpha_m^l(\boldsymbol{e}_j)) - (\omega_j^l(\boldsymbol{e}_m) - \alpha_j^l(\boldsymbol{e}_m)) = -(\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) + (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)),
\end{aligned}$$

which conclude that $\omega_l^j = \alpha_l^j$. Summing up, we have

$$d\boldsymbol{e}_j = c\omega^j \boldsymbol{e}_0 + \sum_s \omega_j^s \boldsymbol{e}_s. \tag{6.7}$$

Then the frame $\mathcal{F}$ in (6.5) satisfies

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \text{where} \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\boldsymbol{\omega}^T \\ c\boldsymbol{\omega} & \Omega \end{pmatrix} \quad \text{and} \quad \boldsymbol{\omega} := (\omega^1, \dots, \omega^n)^T. \tag{6.8}$$

The integrability condition of (6.8) is

$$O = d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = \begin{pmatrix} c^2\boldsymbol{\omega}^T \wedge \boldsymbol{\omega} & c\left(d\boldsymbol{\omega}^T + \omega^T \wedge \Omega\right) \\ c\left(d\boldsymbol{\omega} + \Omega \wedge \boldsymbol{\omega}\right) & d\Omega + \Omega \wedge \Omega + c^2\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T \end{pmatrix}.$$

The lower-right components of the identity above yields

$$\kappa_i^j + c^2\omega^i \wedge \omega^j = 0.$$

Hence the sectional curvature of $(H^n(-c^2), g_H) = -c^2$. □

Remark 6.6. One can show the completeness of $(H^n(-c^2), g_H)$ (cf. MTH.B505). Hence the hyperbolic space is a simply connected space form of constant negative sectional curvature.

6.4 Isometries

A C^∞-map $f\colon M \to N$ between manifolds M and N induces a linear map

$$(df)_p\colon T_pM \ni X \longmapsto (df)_p(X) = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t) \in T_{f(p)}N,$$

where $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = X$, called the *differential* of f. Since $p \in M$ is arbitrary, this induces a bundle homomorphism $df\colon TM \to TN$.

Definition 6.7. A *vector field on N along a smooth map $f\colon M \to N$* is a map $X\colon M \to TN$ satisfying $\pi \circ X = f$, where $\pi\colon TN \to N$ is the canonical projection.

Then for each vector field $X \in \mathfrak{X}(M)$, $df(X)$ is a vector field on N along f.

Definition 6.8. A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M,g) and (N,h) is called a *local isometry* if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X,Y) := h(df(X), df(Y)) = g(X,Y)$$

holds for X, $Y \in T_pM$ and $p \in M$.

Lemma 6.9. *A local isometry is an immersion.*

Proof. Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be a (local) orthonormal frame of M, where $n = \dim M$. Set $\boldsymbol{v}_j := df(\boldsymbol{e}_j)$ $(j = 1,\dots,n)$ for a smooth map $f\colon (M,g) \to (N,h)$. If f is a local isometry, $[\boldsymbol{v}_1(p),\dots,\boldsymbol{v}_n(p)]$ is an orthonormal system in $T_{f(p)}N$, because

$$h(\boldsymbol{v}_i,\boldsymbol{v}_j) = h(df(\boldsymbol{e}_i), df(\boldsymbol{e}_j)) = f^*h(\boldsymbol{e}_i,\boldsymbol{e}_j) = g(\boldsymbol{e}_i,\boldsymbol{e}_j).$$

Hence the differential $(df)_p$ is of rank n. □

The proof of Lemma 6.9 suggests the following fact:

Corollary 6.10. *A smooth map $f\colon (M,g) \to (N,h)$ is a local isometry if and only if for each $p \in M$,*

$$[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n] := [df(\boldsymbol{e}_1),\dots,df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

6.5 Local uniqueness of space forms

Theorem 6.11. *Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U,g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where*

$$N^n(k) = \begin{cases} S^n(k) & (k > 0) \\ \mathbb{R}^n & (k = 0) \\ H^n(k) & (k < 0). \end{cases}$$

Proof. Take an orthonormal frame $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ on U, and let (ω^j), $\varOmega = (\omega_i^j)$ and $K = (\kappa_i^j)$ be the dual frame, the connection form, and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively. Since the sectional curvature is constant k, $\kappa_i^j = k\omega^i \wedge \omega^j$ holds for each (i,j), because of Theorem 6.1.

First, consider the case $k = 0$: In this case, $K = d\varOmega + \varOmega \wedge \varOmega = O$, and then by Theorem 4.5, there exists the unique matrix valued function $\mathcal{F}\colon U \to \mathrm{SO}(n)$ satisfying

$$d\mathcal{F} = \mathcal{F}\varOmega, \qquad \mathcal{F}(p_0) = \mathrm{id},$$

where $p_0 \in U$ is a fixed point. Decompose the matrix $\mathcal{F}$ into column vectors as $\mathcal{F} = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]$, and define an $\mathbb{R}^n$-valued one form

$$\boldsymbol{\alpha} := \sum_{j=1}^n \omega^j \boldsymbol{v}_j.$$

Then

$$d\boldsymbol{\alpha} = \sum_{j=1}^n \Bigl(d\omega^j \boldsymbol{v}_j - \omega^j \wedge d\boldsymbol{v}_j\Bigr) = \sum_{j,s} \Bigl(\omega^s \wedge \omega_s^j\Bigr)\boldsymbol{v}_j - \sum_{j,s} \Bigl(\omega^j \wedge \omega_j^s\Bigr)\boldsymbol{v}_s = \boldsymbol{0}.$$

Hence by the Poincaré lemma (Theorem 4.8), there exists a smooth map $f\colon U \to \mathbb{R}^n$ satisfying $df = \boldsymbol{\alpha}$. For such an f, it holds that

$$df(\boldsymbol{e}_s) = \alpha(\boldsymbol{e}_s) = \sum_{j=1}^{n} \omega^j(\boldsymbol{e}_s)\boldsymbol{v}_j = \boldsymbol{v}_s$$

for $s = 1, \dots, n$. Hence $[df(\boldsymbol{e}_1), \dots, df(\boldsymbol{e}_n)] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ is an orthonormal frame, and then f is a local isometry because Corollary 6.10.

Next, consider the case $k = -c^2 < 0$. We set

$$\widetilde{\Omega} := \begin{pmatrix} 0 & c\boldsymbol{\omega}^T \\ c\boldsymbol{\omega} & \Omega \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$$

as in (6.8) in Section **??**. Since $\kappa_i^j = k\omega^i \wedge \omega^j = -c^2\omega^i \wedge \omega^j$, $d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = O$ holds as seen in Section **??**. Hence there exists an matrix valued function $\mathcal{F}\colon U \to \mathrm{M}_{n+1}(\mathbb{R})$ satisfying

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \mathcal{F}(p_0) = \mathrm{id}, \tag{6.9}$$

where $p_0 \in U$ is a fixed point. Notice that

$$\widetilde{\Omega}^T Y + Y\widetilde{\Omega} = O \qquad Y = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

holds,

$$d(\mathcal{F}Y\mathcal{F}^T) = \mathcal{F}\widetilde{\Omega}Y\mathcal{F}^T + \mathcal{F}Y\widetilde{\Omega}^T\mathcal{F}^T = \mathcal{F}(\widetilde{\Omega}Y + Y\widetilde{\Omega}^T)\mathcal{F}^T = O.$$

Hence, by the initial condition,

$$\mathcal{F}Y\mathcal{F}^T = Y, \qquad \text{that is,} \qquad (\mathcal{F}Y)^{-1} = \mathcal{F}^T Y.$$

Thus, we have

$$\mathcal{F}^T Y\mathcal{F} = (\mathcal{F}Y)^{-1}\mathcal{F} = Y\mathcal{F}^{-1}\mathcal{F} = Y. \tag{6.10}$$

Decompose $\mathcal{F} = [\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$. Then (6.10) is equivalent to

$$-\langle \boldsymbol{v}_0, \boldsymbol{v}_0 \rangle_L = \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle_L = \dots = \langle \boldsymbol{v}_n, \boldsymbol{v}_n \rangle_L = 1, \qquad \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0 \quad (\text{if } i \neq j). \tag{6.11}$$

In particular, the 0-th component of $\boldsymbol{v}_0$ never vanishes, since

$$-1 = \langle \boldsymbol{v}_0, \boldsymbol{v}_0 \rangle_L = -(v_0^0)^2 + (v_0^1)^2 + \dots + (v_0^n)^2 \qquad \boldsymbol{v}_0 = (v_0^0, v_0^1, \dots, v_0^n)^T.$$

Moreover, by the initial condition $\boldsymbol{v}_0(p_0) = (1, 0, \dots, 0)^T$,

$$v_0^0 > 0 \tag{6.12}$$

holds.

Set $f := \frac{1}{c}\boldsymbol{v}_0$. Then $f\colon U \to \mathbb{R}_1^{n+1}$ is the desired map. In fact, by (6.11) and (6.12),

$$f \in H^n(-c^2) = \left\{ \boldsymbol{x} = (x^0, \dots, x^n)^T \in \mathbb{R}_1^{n+1} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -\frac{1}{c^2}, cx^0 > 0 \right\},$$

and

$$df(\boldsymbol{e}_j) = \frac{1}{c} d\boldsymbol{v}_0(\boldsymbol{e}_j) = \sum_{s=1}^{n} \omega^s(\boldsymbol{e}_j)\boldsymbol{v}_s = \boldsymbol{v}_j.$$

Hence $[\boldsymbol{v}_j] = [\boldsymbol{e}_j]$ is an orthonormal frame because (6.11).

The case $k > 0$ is left as an exercise. □

Exercises

6-1 Prove that the sphere

$$S^3(1) = \{\boldsymbol{x} \in \mathbb{R}^4 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

of radius 1 in the Euclidean 4-space is of constant sectional curvature 1.

6-2 Prove Theorem 6.11 for $k = 1$ and $n = 2$, assuming Exercise 6-1.