1 Linear Ordinary Differential Equations

The fundamental theorem for ordinary differential equations. Consider a function

$$\boldsymbol{f}\colon I\times U\ni(t,\boldsymbol{x})\longmapsto\boldsymbol{f}(t,\boldsymbol{x})\in\mathbb{R}^m \tag{1.1}$$

of class C^1, where $I\subset\mathbb{R}$ is an interval and $U\subset\mathbb{R}^m$ is a domain in the Euclidean space $\mathbb{R}^m$. For any fixed $t_0\in I$ and $\boldsymbol{x}_0\in U$, the condition

$$\frac{d}{dt}\boldsymbol{x}(t)=\boldsymbol{f}\bigl(t,\boldsymbol{x}(t)\bigr),\qquad\boldsymbol{x}(t_0)=\boldsymbol{x}_0 \tag{1.2}$$

of an $\mathbb{R}^m$-valued function $t\mapsto\boldsymbol{x}(t)$ is called the *initial value problem of ordinary differential equation* for unknown function $\boldsymbol{x}(t)$. A function $\boldsymbol{x}\colon I\to U$ satisfying (1.2) is called a *solution* of the initial value problem.

Fact 1.1 (The existence theorem for ODE's)**.** *Let $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ be a C^1-function as in* (1.1)*. Then, for any $\boldsymbol{x}_0\in U$ and $t_0\in I$, there exists a positive number ε and a C^1-function $\boldsymbol{x}\colon I\cap(t_0-\varepsilon,t_0+\varepsilon)\to U$ satisfying* (1.2)*.*

Consider two solutions $\boldsymbol{x}_j\colon J_j\to U$ $(j=1,2)$ of (1.2) defined on subintervals $J_j\subset I$ containing t_0. Then the function $\boldsymbol{x}_2$ is said to be an *extension* of $\boldsymbol{x}_1$ if $J_1\subset J_2$ and $\boldsymbol{x}_2|_{J_1}=\boldsymbol{x}_1$. A solution $\boldsymbol{x}$ of (1.2) is said to be *maximal* if there are no non-trivial extension of it.

Fact 1.2 (The uniqueness for ODE's)**.** *The maximal solution of* (1.2) *is unique.*

Fact 1.3 (Smoothness of the solutions)**.** *If $\boldsymbol{f}\colon I\times U\to\mathbb{R}^m$ is of class C^r $(r=1,\dots,\infty)$, the solution of* (1.2) *is of class C^{r+1}. Here, $\infty+1=\infty$, as a convention.*

Let $V\subset\mathbb{R}^k$ be another domain of $\mathbb{R}^k$ and consider a C^∞-function

$$\boldsymbol{h}\colon I\times U\times V\ni(t,\boldsymbol{x};\boldsymbol{\alpha})\mapsto\boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})\in\mathbb{R}^m. \tag{1.3}$$

For fixed $t_0\in I$, we denote by $\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ the (unique, maximal) solution of (1.2) for $\boldsymbol{f}(t,\boldsymbol{x})=\boldsymbol{h}(t,\boldsymbol{x};\boldsymbol{\alpha})$. Then

Fact 1.4. *The map $(t,\boldsymbol{x}_0;\boldsymbol{\alpha})\mapsto\boldsymbol{x}(t;\boldsymbol{x}_0,\boldsymbol{\alpha})$ is of class C^∞.*

Example 1.5. (1) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=\lambda x$, where λ is a constant. Then $x(t)=x_0\exp(\lambda t)$ defined on $\mathbb{R}$ is the maximal solution to

$$\frac{d}{dt}x(t)=f(t,x(t))=\lambda x(t),\qquad x(0)=x_0.$$

(2) Let $m=2$, $I=\mathbb{R}$, $U=\mathbb{R}^2$ and $\boldsymbol{f}(t;(x,y))=(y,-\omega^2x)$, where ω is a constant. Then

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}x_0\cos\omega t+\frac{y_0}{\omega}\sin\omega t\\-x_0\omega\sin\omega t+y_0\cos\omega t\end{pmatrix}$$

is the unique solution of

$$\frac{d}{dt}\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}y(t)\\-\omega^2x(t)\end{pmatrix},\qquad\begin{pmatrix}x(0)\\y(0)\end{pmatrix}=\begin{pmatrix}x_0\\y_0\end{pmatrix},$$

defined on $\mathbb{R}$. This differential equation can be considered a single equation

$$\frac{d^2}{dt^2}x(t)=-\omega^2x(t),\quad x(0)=x_0,\quad\frac{dx}{dt}(0)=y_0$$

of order 2.

(3) Let $m=1$, $I=\mathbb{R}$, $U=\mathbb{R}$ and $f(t,x)=1+x^2$. Then $x(t)=\tan t$ defined on $(-\frac{\pi}{2},\frac{\pi}{2})$ is the unique maximal solution of the initial value problem

$$\frac{dx}{dt}=1+x^2,\qquad x(0)=0.$$

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Linear Ordinary Differential Equations. The ordinary differential equation (1.2) is said to be *linear* if the function (1.1) is a linear function in $\boldsymbol{x}$, that is, a linear differential equation is in a form

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

where $A(t)$ and $\boldsymbol{b}(t)$ are $m \times m$-matrix-valued and $\mathbb{R}^m$-valued functions in t.

For the sake of later use, we consider, in this lecture, the special form of linear differential equation for matrix-valued unknown functions as follows: Let $\mathrm{M}_n(\mathbb{R})$ be the set of $n \times n$-matrices with real components, and take functions

$$\Omega\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}), \quad \text{and} B\colon I \longrightarrow \mathrm{M}_n(\mathbb{R}),$$

where $I \subset \mathbb{R}$ is an interval. Identifying $\mathrm{M}_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$, we assume Ω and B are continuous functions (with respect to the topology of $\mathbb{R}^{n^2} = \mathrm{M}_n(\mathbb{R})$). Then we can consider the linear ordinary differential equation for matrix-valued unknown $X(t)$ as

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0, \tag{1.4}$$

where X_0 is given constant matrix.

Then, the fundamental theorem of *linear* ordinary equation states that *the maximal solution of* (1.4) *is defined on whole* I. To prove this, we prepare some materials related to matrix-valued functions.

Preliminaries: Matrix Norms. Denote by $\mathrm{M}_n(\mathbb{R})$ the set of $n \times n$-matrices with real components, which can be identified the vector space $\mathbb{R}^{n^2}$. In particular, the Euclidean norm of $\mathbb{R}^{n^2}$ induces a norm

$$|X|_{\mathrm{E}} = \sqrt{\operatorname{tr}(X^T X)} = \sqrt{\sum_{i,j=1}^{n} x_{ij}^2} \tag{1.5}$$

on $\mathrm{M}_n(\mathbb{R})$. On the other hand, we let

$$|X|_{\mathrm{M}} := \sup\left\{\frac{|X\boldsymbol{v}|}{|\boldsymbol{v}|}\,;\,\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}\right\}, \tag{1.6}$$

where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$.

Lemma 1.6. (1) *The map* $X \mapsto |X|_{\mathrm{M}}$ *is a norm of* $\mathrm{M}_n(\mathbb{R})$.

(2) *For* X, $Y \in \mathrm{M}_n(\mathbb{R})$, *it holds that* $|XY|_{\mathrm{M}} \leqq |X|_{\mathrm{M}}\,|Y|_{\mathrm{M}}$.

(3) *Let* $\lambda = \lambda(X)$ *be the maximum eigenvalue of semi-positive definite symmetric matrix* $X^T X$. *Then* $|X|_{\mathrm{M}} = \sqrt{\lambda}$ *holds.*

(4) $(1/\sqrt{n})|X|_{\mathrm{E}} \leqq |X|_{\mathrm{M}} \leqq |X|_{\mathrm{E}}$.

(5) *The map* $|\cdot|_{\mathrm{M}}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ *is continuous with respect to the Euclidean norm.*

Proof. Since $|X\boldsymbol{v}|/|\boldsymbol{v}|$ is invariant under scalar multiplications to $\boldsymbol{v}$, we have $|X|_{\mathrm{M}} = \sup\{|X\boldsymbol{v}|\,;\,\boldsymbol{v} \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Since $S^{n-1} \ni \boldsymbol{x} \mapsto |A\boldsymbol{x}| \in \mathbb{R}$ is a continuous function defined on a compact space, it takes the maximum. Thus, the right-hand side of (1.6) is well-defined. It is easy to verify that $|\cdot|_{\mathrm{M}}$ satisfies the axiom of the norm[1].

[1] $|X|_{\mathrm{M}} > 0$ whenever $X \neq O$, $|\alpha X|_{\mathrm{M}} = |\alpha|\,|X|_{\mathrm{M}}$, and the triangle inequality.

Since $A := X^TX$ is positive semi-definite, the eigenvalues λ_j $(j = 1, \dots, n)$ are non-negative real numbers. In particular, there exists an orthonormal basis $[\boldsymbol{a}_j]$ of $\mathbb{R}^n$ satisfying $A\boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j$ $(j = 1, \dots, n)$. Let λ be the maximum eigenvalue of A, and write $\boldsymbol{v} = v_1\boldsymbol{a}_1 + \dots + v_n\boldsymbol{a}_n$. Then it holds that

$$\langle X\boldsymbol{v}, X\boldsymbol{v}\rangle = \lambda_1 v_1^2 + \dots + \lambda_n v_n^2 \leqq \lambda \langle \boldsymbol{v}, \boldsymbol{v}\rangle\,,$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product of $\mathbb{R}^n$. The equality of this inequality holds if and only if $\boldsymbol{v}$ is the λ-eigenvector, proving (3). Noticing the norm (1.5) is invariant under conjugations $X \mapsto P^TXP$ $(P \in \mathrm{O}(n))$, we obtain $|X|_{\mathrm{E}} = \sqrt{\lambda_1^2 + \dots + \lambda_n^2}$ by diagonalizing X^TX by an orthogonal matrix P. Then we obtain (4). Hence two norms $|\cdot|_{\mathrm{E}}$ and $|\cdot|_{\mathrm{M}}$ induce the same topology as $\mathrm{M}_n(\mathbb{R})$. In particular, we have (5). □

Preliminaries: Matrix-valued Functions.

Lemma 1.7. *Let X and Y be C^∞-maps defined on a domain $U \subset \mathbb{R}^m$ into $\mathrm{M}_n(\mathbb{R})$. Then*

(1) $\dfrac{\partial}{\partial u_j}(XY) = \dfrac{\partial X}{\partial u_j}Y + X\dfrac{\partial Y}{\partial u_j}$,

(2) $\dfrac{\partial}{\partial u_j}\det X = \operatorname{tr}\left(\widetilde{X}\dfrac{\partial X}{\partial u_j}\right)$, *and*

(3) $\dfrac{\partial}{\partial u_j}X^{-1} = -X^{-1}\dfrac{\partial X}{\partial u_j}X^{-1}$,

where $\widetilde{X}$ is the cofactor matrix of X, and we assume in (3) that X is a regular matrix.

Proof. The formula (1) holds because the definition of matrix multiplication and the Leibnitz rule, Denoting $' = \partial/\partial u_j$,

$$O = (\mathrm{id})' = (X^{-1}X)' = (X^{-1})X' + (X^{-1})'X$$

implies (3), where id is the identity matrix.

Decompose the matrix X into column vectors as $X = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$. Since the determinant is multi-linear form for n-tuple of column vectors, it holds that

$$(\det X)' = \det(\boldsymbol{x}_1', \boldsymbol{x}_2, \dots, \boldsymbol{x}_n) + \det(\boldsymbol{x}_1, \boldsymbol{x}_2', \dots, \boldsymbol{x}_n) + \dots + \det(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n').$$

Then by cofactor expansion of the right-hand side, we obtain (2). □

Proposition 1.8. *Assume two C^∞ matrix-valued functions $X(t)$ and $\Omega(t)$ satisfy*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0. \tag{1.7}$$

Then

$$\det X(t) = (\det X_0)\exp\int_{t_0}^{t}\operatorname{tr}\Omega(\tau)\,d\tau \tag{1.8}$$

holds. In particular, if $X_0 \in \mathrm{GL}(n,\mathbb{R})$,[2] *then $X(t) \in \mathrm{GL}(n,\mathbb{R})$ for all t.*

Proof. By (2) of Lemma 1.7, we have

$$\begin{aligned}\frac{d}{dt}\det X(t) &= \operatorname{tr}\left(\widetilde{X}(t)\frac{dX(t)}{dt}\right) = \operatorname{tr}\left(\widetilde{X}(t)X(t)\Omega(t)\right)\\ &= \operatorname{tr}\bigl(\det X(t)\Omega(t)\bigr) = \det X(t)\operatorname{tr}\Omega(t).\end{aligned}$$

Here, we used the relation $\widetilde{X}X = X\widetilde{X} = (\det X)\,\mathrm{id}$[3]. Hence $\frac{d}{dt}\bigl(\rho(t)^{-1}\det X(t)\bigr) = 0$, where $\rho(t)$ is the right-hand side of (1.8). □

[2] $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\,\det A \neq 0\}$: the general linear group.
[3] In this lecture, id denotes the identity matrix.

Corollary 1.9. *If $\Omega(t)$ in (1.7) satisfies $\operatorname{tr}\Omega(t) = 0$, $\det X(t)$ is constant. In particular, if $X_0 \in \mathrm{SL}(n,\mathbb{R})$, X is a function valued in $\mathrm{SL}(n,\mathbb{R})$* [4].

Proposition 1.10. *Assume $\Omega(t)$ in (1.7) is skew-symmetric for all t, that is, $\Omega^T + \Omega$ is identically O. If $X_0 \in \mathrm{O}(n)$ (resp. $X_0 \in \mathrm{SO}(n)$)*[5]*, then $X(t) \in \mathrm{O}(n)$ (resp. $X(t) \in \mathrm{SO}(n)$) for all t.*

Proof. By (1) in Lemma 1.7,

$$\begin{aligned}\frac{d}{dt}(XX^T) &= \frac{dX}{dt}X^T + X\left(\frac{dX}{dt}\right)^T \\ &= X\Omega X^T + X\Omega^T X^T = X(\Omega + \Omega^T)X^T = O.\end{aligned}$$

Hence XX^T is constant, that is, if $X_0 \in \mathrm{O}(n)$,

$$X(t)X(t)^T = X(t_0)X(t_0)^T = X_0X_0^T = \mathrm{id}\,.$$

If $X_0 \in \mathrm{O}(n)$, this proves the first case of the proposition. Since $\det A = \pm 1$ when $A \in \mathrm{O}(n)$, the second case follows by continuity of $\det X(t)$. □

Preliminaries: Norms of Matrix-Valued functions. Let $I = [a, b]$ be a closed interval, and denote by $C^0(I, \mathrm{M}_n(\mathbb{R}))$ the set of continuous functions $X : I \to \mathrm{M}_n(\mathbb{R})$. For any positive number k, we define

$$\|X\|_{I,k} := \sup\left\{e^{-kt}|X(t)|_{\mathrm{M}}\,;\, t \in I\right\} \tag{1.9}$$

for $X \in C^0(I, \mathrm{M}_n(\mathbb{R}))$. When $k = 0$, $\|\cdot\|_{I,0}$ is the *uniform norm* for continuous functions, which is complete. Similarly, one can prove the following in the same way:

Lemma 1.11. *The norm $\|\cdot\|_{I,k}$ on $C^0(I, \mathrm{M}_n(\mathbb{R}))$ is complete.*

Linear Ordinary Differential Equations. We prove the fundamental theorem for *linear* ordinary differential equations.

Proposition 1.12. *Let $\Omega(t)$ be a C^∞-function valued in $\mathrm{M}_n(\mathbb{R})$ defined on an interval I. Then for each $t_0 \in I$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,\mathrm{id}}(t)$ such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,. \tag{1.10}$$

Proof. Uniqueness: Assume $X(t)$ and $Y(t)$ satisfy (1.10). Then

$$Y(t) - X(t) = \int_{t_0}^{t} \bigl(Y'(\tau) - X'(\tau)\bigr)\,d\tau = \int_{t_0}^{t} \bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\,d\tau \qquad \left({}' = \frac{d}{dt}\right)$$

holds. Hence for an arbitrary closed interval $J \subset I$,

$$\begin{aligned}|Y(t) - X(t)|_{\mathrm{M}} &\leqq \left|\int_{t_0}^{t} \left|\bigl(Y(\tau) - X(\tau)\bigr)\Omega(\tau)\right|_{\mathrm{M}} d\tau\right| \leqq \left|\int_{t_0}^{t} |Y(\tau) - X(\tau)|_{\mathrm{M}}\, |\Omega(\tau)|_{\mathrm{M}}\, d\tau\right| \\ &= \left|\int_{t_0}^{t} e^{-k\tau}\, |Y(\tau) - X(\tau)|_{\mathrm{M}}\, e^{k\tau}\, |\Omega(\tau)|_{\mathrm{M}}\, d\tau\right| \leqq \|Y - X\|_{J,k} \sup_J |\Omega|_{\mathrm{M}} \left|\int_{t_0}^{t} e^{k\tau}\,d\tau\right| \\ &= \|Y - X\|_{J,k} \frac{\sup_J |\Omega|_{\mathrm{M}}}{|k|} e^{kt} \left|1 - e^{-k(t-t_0)}\right| \leqq \|Y - X\|_{J,k} \sup_J |\Omega|_{\mathrm{M}} \frac{e^{kt}}{|k|}\end{aligned}$$

[4] $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, \det A = 1\}$; the special lienar group.
[5] $\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R})\,;\, A^TA = AA^T = \mathrm{id}\}$: the orthogonal group; $\mathrm{SO}(n) = \{A \in \mathrm{O}(n)\,;\, \det A = 1\}$: the special orthogonal group.

holds for $t \in J$. Thus, for an appropriate choice of $k \in \mathbb{R}$, it holds that

$$||Y - X||_{J,k} \leqq \frac{1}{2}||Y - X||_{J,k},$$

that is, $||Y - X||_{J,k} = 0$, proving $Y(t) = X(t)$ for $t \in J$. Since J is arbitrary, $Y = X$ holds on I.
Existence: Let $J := [t_0, a] \subset I$ be a closed interval, and define a sequence $\{X_j\}$ of matrix-valued functions defined on I satisfying $X_0(t) = \mathrm{id}$ and

$$X_{j+1}(t) = \mathrm{id} + \int_{t_0}^{t} X_j(\tau)\Omega(\tau)\,d\tau \quad (j = 0, 1, 2, \dots). \tag{1.11}$$

Let $k := 2\sup_J |\Omega|_{\mathrm{M}}$. Then

$$\begin{aligned}|X_{j+1}(t) - X_j(t)|_{\mathrm{M}} &\leqq \int_{t_0}^{t} |X_j(\tau) - X_{j-1}(\tau)|_{\mathrm{M}}|\Omega(\tau)|_{\mathrm{M}}\,d\tau \\ &\leqq \frac{e^{k(t-t_0)}}{|k|}\sup_J |\Omega|_{\mathrm{M}}||X_j - X_{j-1}||_{J,k}\end{aligned}$$

for an appropriate choice of $k \in \mathbb{R}$, and hence $||X_{j+1} - X_j||_{J,k} \leqq \frac{1}{2}||X_j - X_{j-1}||_{J,k}$, that is, $\{X_j\}$ is a Cauchy sequence with respect to $||\cdot||_{J,k}$. Thus, by completeness (Lemma 1.11), it converges to some $X \in C^0(J, \mathrm{M}_n(\mathbb{R}))$. By (1.11), the limit X satisfies

$$X(t_0) = \mathrm{id}, \qquad X(t) = \mathrm{id} + \int_{t_0}^{t} X(\tau)\Omega(\tau)\,d\tau.$$

Applying the fundamental theorem of calculus, we can see that X satisfies $X'(t) = X(t)\Omega(t)$ ($' = d/dt$). Since J can be taken arbitrarily, existence of the solution on I is proven.

Finally, we shall prove that X is of class C^∞. Since $X'(t) = X(t)\Omega(t)$, the derivative X' of X is continuous. Hence X is of class C^1, and so is $X(t)\Omega(t)$. Thus we have that $X'(t)$ is of class C^1, and then X is of class C^2. Iterating this argument, we can prove that $X(t)$ is of class C^r for arbitrary r. □

Corollary 1.13. *Let $\Omega(t)$ be a matrix-valued C^∞-function defined on an interval I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X_{t_0,X_0}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \bigl(X(t) := X_{t_0,X_0}(t)\bigr) \tag{1.12}$$

In particular, $X_{t_0,X_0}(t)$ is of class C^∞ in X_0 and t.

Proof. We rewrite $X(t)$ in Proposition 1.12 as $Y(t) = X_{t_0,\mathrm{id}}(t)$. Then the function

$$X(t) := X_0 Y(t) = X_0 X_{t_0,\mathrm{id}}(t), \tag{1.13}$$

is desired one. Conversely, assume $X(t)$ satisfies the conclusion. Noticing $Y(t)$ is a regular matrix for all t because of Proposition 1.8,

$$W(t) := X(t)Y(t)^{-1}$$

satisfies

$$\frac{dW}{dt} = \frac{dX}{dt}Y^{-1} - XY^{-1}\frac{dY}{dt}Y^{-1} = X\Omega Y^{-1} - XY^{-1}Y\Omega Y^{-1} = O.$$

Hence

$$W(t) = W(t_0) = X(t_0)Y(t_0)^{-1} = X_0.$$

Hence the uniqueness is obtained. The final part is obvious by the expression (1.13). □

Proposition 1.14. *Let $\Omega(t)$ and $B(t)$ be matrix-valued C^∞-functions defined on I. Then for each $t_0 \in I$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function defined on I satisfying*

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0. \tag{1.14}$$

Proof. Rewrite X in Proposition 1.12 as $Y := X_{t_0,\mathrm{id}}$. Then

$$X(t) = \left(X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau\right)Y(t) \tag{1.15}$$

satisfies (1.14). Conversely, if X satisfies (1.14), $W := XY^{-1}$ satisfies

$$X' = W'Y + WY' = W'Y + WY\Omega, \quad X\Omega + B = WY\Omega + B,$$

and then we have $W' = BY^{-1}$. Since $W(t_0) = X_0$,

$$W = X_0 + \int_{t_0}^{t} B(\tau)Y^{-1}(\tau)\,d\tau.$$

Thus we obtain (1.15). □

Theorem 1.15. *Let I and U be an interval and a domain in $\mathbb{R}^m$, respectively, and let $\Omega(t,\boldsymbol{\alpha})$ and $B(t,\boldsymbol{\alpha})$ be matrix-valued C^∞-functions defined on $I \times U$ ($\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_m)$). Then for each $t_0 \in I$, $\boldsymbol{\alpha} \in U$ and $X_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique matrix-valued C^∞-function $X(t) = X_{t_0,X_0,\boldsymbol{\alpha}}(t)$ defined on I such that*

$$\frac{dX(t)}{dt} = X(t)\Omega(t,\boldsymbol{\alpha}) + B(t,\boldsymbol{\alpha}), \qquad X(t_0) = X_0. \tag{1.16}$$

Moreover,

$$I \times I \times \mathrm{M}_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0,X_0,\boldsymbol{\alpha}}(t) \in \mathrm{M}_n(\mathbb{R})$$

is a C^∞-map.

Proof. Let $\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) := \Omega(t+t_0,\boldsymbol{\alpha})$ and $\widetilde{B}(t,\tilde{\boldsymbol{\alpha}}) = B(t+t_0,\boldsymbol{\alpha})$, and let $\widetilde{X}(t) := X(t+t_0)$. Then (1.16) is equivalent to

$$\frac{d\widetilde{X}(t)}{dt} = \widetilde{X}(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}}) + \widetilde{B}(t,\tilde{\boldsymbol{\alpha}}), \quad \widetilde{X}(0) = X_0, \tag{1.17}$$

where $\tilde{\boldsymbol{\alpha}} := (t_0,\alpha_1,\dots,\alpha_m)$. There exists the unique solution $\widetilde{X}(t) = \widetilde{X}_{0,X_0,\tilde{\boldsymbol{\alpha}}}(t)$ of (1.17) for each $\tilde{\boldsymbol{\alpha}}$ because of Proposition 1.14. So it is sufficient to show differentiability with respect to the parameter $\tilde{\boldsymbol{\alpha}}$. We set $Z = Z(t)$ the unique solution of

$$\frac{dZ}{dt} = Z\widetilde{\Omega} + \widetilde{X}\frac{\partial\widetilde{\Omega}}{\partial\alpha_j} + \frac{\partial\widetilde{B}}{\partial\alpha_j}, \qquad Z(0) = O. \tag{1.18}$$

Then it holds that $Z = \partial\widetilde{X}/\partial\alpha_j$. In particular, by the proof of Proposition 1.14, it holds that

$$Z = \frac{\partial\widetilde{X}}{\partial\alpha_j} = \left(\int_0^t \left(\widetilde{X}(\tau)\frac{\partial\widetilde{\Omega}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j} + \frac{\partial\widetilde{B}(\tau,\tilde{\boldsymbol{\alpha}})}{\partial\alpha_j}\right)Y^{-1}(\tau)d\tau\right)Y(t).$$

Here, $Y(t)$ is the unique matrix-valued C^∞-function satisfying $Y'(t) = Y(t)\widetilde{\Omega}(t,\tilde{\boldsymbol{\alpha}})$, and $Y(0) = \mathrm{id}$. Hence $\widetilde{X}$ is a C^∞-function in $(t,\tilde{\boldsymbol{\alpha}})$. □

Fundamental Theorem for Space Curves. As an application, we prove the fundamental theorem for space curves. A C^∞-map $\gamma\colon I \to \mathbb{R}^3$ defined on an interval $I \subset \mathbb{R}$ into $\mathbb{R}^3$ is said to be a *regular curve* if $\dot\gamma \neq \mathbf{0}$ holds on I. For a regular curve $\gamma(t)$, there exists a parameter change $t = t(s)$ such that $\tilde\gamma(s) := \gamma(t(s))$ satisfies $|\tilde\gamma'(s)| = 1$. Such a parameter s is called the *arc-length parameter*.

Let $\gamma(s)$ be a regular curve in $\mathbb{R}^3$ parametrized by the arc-length satisfying $\gamma''(s) \neq \mathbf{0}$ for all s. Then

$$\boldsymbol{e}(s) := \gamma'(s), \qquad \boldsymbol{n}(s) := \frac{\gamma''(s)}{|\gamma''(s)|}, \qquad \boldsymbol{b}(s) := \boldsymbol{e}(s) \times \boldsymbol{n}(s)$$

forms a positively oriented orthonormal basis $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ of $\mathbb{R}^3$ for each s. Regarding each vector as column vector, we have the matrix-valued function

$$\mathcal{F}(s) := (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) \in \mathrm{SO}(3). \tag{1.19}$$

in s, which is called the *Frenet frame* associated to the curve γ. Under the situation above, we set

$$\kappa(s) := |\gamma''(s)| > 0, \qquad \tau(s) := -\left\langle \boldsymbol{b}'(s), \boldsymbol{n}(s) \right\rangle,$$

which are called the *curvature* and *torsion*, respectively, of γ. Using these quantities, the Frenet frame satisfies

$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\varOmega, \qquad \varOmega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{1.20}$$

Proposition 1.16. *The curvature and the torsion are invariant under the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ of $\mathbb{R}^3$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$). Conversely, two curves $\gamma_1(s)$, $\gamma_2(s)$ parametrized by arc-length parameter have common curvature and torsion, there exist $A \in \mathrm{SO}(3)$ and $\boldsymbol{b} \in \mathbb{R}^3$ such that $\gamma_2 = A\gamma_1 + \boldsymbol{b}$.*

Proof. Let κ, τ and $\mathcal{F}_1$ be the curvature, torsion and the Frenet frame of γ_1, respectively. Then the Frenet frame of $\gamma_2 = A\gamma_1 + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) is $\mathcal{F}_2 = A\mathcal{F}_1$. Hence both $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (1.20), and then γ_1 and γ_2 have common curvature and torsion.

Conversely, assume γ_1 and γ_2 have common curvature and torsion. Then the frenet frame $\mathcal{F}_1$, $\mathcal{F}_2$ both satisfy (1.20). Let $\mathcal{F}$ be the unique solution of (1.20) with $\mathcal{F}(t_0) = \mathrm{id}$. Then by the proof of Corollary 1.13, we have $\mathcal{F}_j(t) = \mathcal{F}_j(t_0)\mathcal{F}(t)$ ($j = 1, 2$). In particular, since $\mathcal{F}_j \in \mathrm{SO}(3)$, $\mathcal{F}_2(t) = A\mathcal{F}_1(t)$ ($A := \mathcal{F}_2(t_0)\mathcal{F}_1(t_0)^{-1} \in \mathrm{SO}(3)$). Comparing the first column of these, $\gamma_2'(s) = A\gamma_1'(t)$ holds. Integrating this, the conclusion follows. □

Theorem 1.17 (The fundamental theorem for space curves)**.**
Let $\kappa(s)$ and $\tau(s)$ be C^∞-functions defined on an interval I satisfying $\kappa(s) > 0$ on I. Then there exists a space curve $\gamma(s)$ parametrized by arc-length whose curvature and torsion are κ and τ, respectively. Moreover, such a curve is unique up to transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$ ($A \in \mathrm{SO}(3)$, $\boldsymbol{b} \in \mathbb{R}^3$) of $\mathbb{R}^3$.

Proof. We have already shown the uniqueness in Proposition 1.16. We shall prove the existence: Let $\varOmega(s)$ be as in (1.20), and $\mathcal{F}(s)$ the solution of (1.20) with $\mathcal{F}(s_0) = \mathrm{id}$. Since $\varOmega$ is skew-symmetric, $\mathcal{F}(s) \in \mathrm{SO}(3)$ by Proposition 1.10. Denoting the column vectors of $\mathcal{F}$ by $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$, and let

$$\gamma(s) := \int_{s_0}^{s} \boldsymbol{e}(\sigma)\, d\sigma.$$

Then $\mathcal{F}$ is the Frenet frame of γ, and κ, and τ are the curvature and torsion of γ, respectively. □

Exercises

1-1 Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where b is a real number.

1-2 Find an explicit expression of a space curve $\gamma(s)$ parametrized by the arc-length s, whose curvature κ and torsion τ satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

2 Integrability Conditions

Let $U \subset \mathbb{R}^m$ be a domain of $(\mathbb{R}^m; u^1, \dots, u^m)$ and consider an m-tuple of $n \times n$-matrix valued C^∞-maps

$$\varOmega_j \colon \mathbb{R}^m \supset U \longrightarrow \mathrm{M}_n(\mathbb{R}) \qquad (j = 1, \dots, m). \tag{2.1}$$

In this section, we consider an initial value problem of a system of linear partial differential equations

$$\frac{\partial X}{\partial u^j} = X\varOmega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0, \tag{2.2}$$

where $\mathrm{P}_0 = (u_0^1, \dots, u_0^m) \in U$ is a fixed point, X is an $n \times n$-matrix valued unknown, and $X_0 \in \mathrm{M}_n(\mathbb{R})$.

Proposition 2.1. *If a C^∞-map $X \colon U \to \mathrm{M}_n(\mathbb{R})$ defined on a domain $U \subset \mathbb{R}^m$ satisfies* (4.1) *with $X_0 \in \mathrm{GL}(n, \mathbb{R})$, then $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ for all $\mathrm{P} \in U$. In addition, if $\varOmega_j$ $(j = 1, \dots, m)$ are skew-symmetric and $X_0 \in \mathrm{SO}(n)$, then $X(\mathrm{P}) \in \mathrm{SO}(n)$ holds for all $\mathrm{P} \in U$.*

Proof. Since U is connected, there exists a continuous path $\gamma_0 \colon [0, 1] \to U$ such that $\gamma_0(0) = \mathrm{P}_0$ and $\gamma_0(1) = \mathrm{P}$. By Whitney's approximation theorem (cf. Theorem 6.21 in [Lee13]), there exists a smooth path $\gamma \colon [0, 1] \to U$ joining P_0 and P approximating γ_0. Since $\hat{X} := X \circ \gamma$ satisfies (2.4) with $\hat{X}(0) = X_0$, Proposition 1.8 yields that $\det \hat{X}(1) \neq 0$ whenever $\det X_0 \neq 0$. Moreover, if $\varOmega_j$'s are skew-symmetric, so is $\varOmega_\gamma(t)$ in (2.4). Thus, by Proposition 1.10, we obtain the latter half of the proposition. □

Proposition 2.2. *If a matrix-valued C^∞ function $X \colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies* (4.1)*, it holds that*

$$\frac{\partial \varOmega_j}{\partial u^k} - \frac{\partial \varOmega_k}{\partial u^j} = \varOmega_j \varOmega_k - \varOmega_k \varOmega_j \tag{2.3}$$

for each (j, k) with $1 \leqq j < k \leqq m$.

Proof. Differentiating (4.1) by u^k, we have

$$\frac{\partial^2 X}{\partial u^k \partial u^j} = \frac{\partial X}{\partial u^k}\varOmega_j + X\frac{\partial \varOmega_j}{\partial u^k} = X\left(\frac{\partial \varOmega_j}{\partial u^k} + \varOmega_k \varOmega_j\right).$$

On the other hand, switching the roles of j and k, we get

$$\frac{\partial^2 X}{\partial u^j \partial u^k} = X\left(\frac{\partial \varOmega_k}{\partial u^j} + \varOmega_j \varOmega_k\right).$$

Since X is of class C^∞, the left-hand sides of these equalities coincide, and so are the right-hand sides. Since $X \in \mathrm{GL}(n, \mathbb{R})$, the conclusion follows. □

The equality (2.3) is called the *integrability condition* or *compatibility condition* of (4.1).

The chain rule yields the following:

Lemma 2.3. *Let $X \colon U \to \mathrm{M}_n(\mathbb{R})$ be a C^∞-map satisfying* (4.1)*. Then for each smooth path $\gamma \colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\varOmega_\gamma(t) \qquad \left(\varOmega_\gamma(t) := \sum_{j=1}^m \varOmega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right) \tag{2.4}$$

on I, where $\gamma(t) = \bigl(u^1(t), \dots, u^m(t)\bigr)$.

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Lemma 2.4. *Let* $\varOmega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* C^∞*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy* (2.3)*. Then for each smooth map*

$$\sigma\colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

defined on a domain $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0, \tag{2.5}$$

where

$$T := \sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial u^j}{\partial t}, \quad W := \sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial u^j}{\partial w} \quad (\widetilde{\varOmega}_j := \varOmega_j \circ \sigma). \tag{2.6}$$

Proof. By the chain rule, we have

$$\begin{aligned}
\frac{\partial T}{\partial w} &= \sum_{j,k=1}^m \frac{\partial \varOmega_j}{\partial u^k} \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} + \sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial^2 u^j}{\partial w \partial t}, \\
\frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \frac{\partial \varOmega_j}{\partial u^k} \frac{\partial u^k}{\partial t} \frac{\partial u^j}{\partial w} + \sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial^2 u^j}{\partial t \partial w} \\
&= \sum_{j,k=1}^m \frac{\partial \varOmega_k}{\partial u^j} \frac{\partial u^j}{\partial t} \frac{\partial u^k}{\partial w} + \sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial^2 u^j}{\partial t \partial w}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} &= \sum_{j,k=1}^m \left(\frac{\partial \varOmega_j}{\partial u^k} - \frac{\partial \varOmega_k}{\partial u^j} \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \sum_{j,k=1}^m \left(\widetilde{\varOmega}_j \widetilde{\varOmega}_k - \widetilde{\varOmega}_k \widetilde{\varOmega}_j \right) \frac{\partial u^k}{\partial w} \frac{\partial u^j}{\partial t} \\
&= \left(\sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial u^j}{\partial t} \right) \left(\sum_{k=1}^m \widetilde{\varOmega}_k \frac{\partial u^k}{\partial w} \right) - \left(\sum_{k=1}^m \widetilde{\varOmega}_k \frac{\partial u^k}{\partial w} \right) \left(\sum_{j=1}^m \widetilde{\varOmega}_j \frac{\partial u^j}{\partial t} \right) \\
&= TW - WT.
\end{aligned}$$

Thus (2.5) holds.

Integrability of linear systems. The main theorem in this section is the following theorem:

Theorem 2.5. *Let* $\varOmega_j\colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* C^∞*-functions defined on a* <u>*simply connected domain*</u> $U \subset \mathbb{R}^m$ *satisfying* (2.3)*. Then for each* $\mathrm{P}_0 \in U$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$*, there exists the unique* $n \times n$*-matrix valued function* $X\colon U \to \mathrm{M}_n(\mathbb{R})$ *satisfying* (4.1)*. Moreover,*

- *if* $X_0 \in \mathrm{GL}(n, \mathbb{R})$*,* $X(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ *holds on* U*,*
- *if* $X_0 \in \mathrm{SO}(n)$ *and* $\varOmega_j$ $(j = 1, \dots, m)$ *are skew-symmetric matrices,* $X \in \mathrm{SO}(n)$ *holds on* U*.*

Proof. The latter half is a direct conclusion of Proposition 2.1. We show the existence of X: Take a smooth path $\gamma\colon [0, 1] \to U$ joining P_0 and P. Then by Theorem 1.15, there exists a unique C^∞-map $\hat{X}\colon [0, 1] \to \mathrm{M}_n(\mathbb{R})$ satisfying (2.4) with initial condition $\hat{X}(0) = X_0$.

We shall show that the value $\hat{X}(1)$ does not depend on choice of paths joining P_0 and P. To show this, choose another smooth path $\tilde{\gamma}$ joining P_0 and P. Since U is simply connected, there exists a homotopy between γ and $\tilde{\gamma}$, that is, there exists a continuous map $\sigma_0\colon [0,1]\times[0,1]\ni (t,w)\mapsto \sigma(t,w)\in U$ satisfying

$$\begin{aligned}\sigma_0(t,0)&=\gamma(t), \qquad \sigma_0(t,1)=\tilde{\gamma}(t),\\ \sigma_0(0,w)&=\mathrm{P}_0, \qquad \sigma_0(1,w)=\mathrm{P}.\end{aligned}\tag{2.7}$$

Then, by Whitney's approximation theorem (Theorem 6.21 in [Lee13]) again, there exists a smooth map $\sigma\colon [0,1]\times[0,1]\to U$ satisfying the same boundary conditions as (2.7):

$$\begin{aligned}\sigma(t,0)&=\gamma(t), \qquad \sigma(t,1)=\tilde{\gamma}(t),\\ \sigma(0,w)&=\mathrm{P}_0, \qquad \sigma(1,w)=\mathrm{P}.\end{aligned}\tag{2.8}$$

We set T and W as in (2.6). For each fixed $w\in[0,1]$, there exists $X_w\colon[0,1]\to \mathrm{M}_n(\mathbb{R})$ such that

$$\frac{dX_w}{dt}(t)=X_w(t)T(t,w), \qquad X_w(0)=X_0.$$

Since $T(t,w)$ is smooth in t and w, the map

$$\check{X}\colon [0,1]\times[0,1]\ni(t,w)\mapsto X_w(t)\in \mathrm{M}_n(\mathbb{R})$$

is a smooth map, because of smoothness in parameter α in Theorem 1.15. To show that $\hat{X}(1)=\check{X}(1,0)$ does not depend on choice of paths, it is sufficient to show that

$$\frac{\partial\check{X}}{\partial w}=\check{X}W\tag{2.9}$$

holds on $[0,1]\times[0,1]$. In fact, by (2.8), $W(1,w)=0$ for all $w\in[0,1]$, and then (2.9) implies that $\check{X}(1,w)$ is constant.

We prove (2.9): By definition, it holds that

$$\frac{\partial\check{X}}{\partial t}=\check{X}T, \qquad \check{X}(0,w)=X_0\tag{2.10}$$

for each $w\in[0,1]$. Hence by (2.5),

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial\check{X}}{\partial w}&=\frac{\partial^2\check{X}}{\partial t\partial w}=\frac{\partial^2\check{X}}{\partial w\partial t}=\frac{\partial}{\partial w}(\check{X}T)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial T}{\partial w}=\frac{\partial\check{X}}{\partial w}T+\check{X}\left(\frac{\partial W}{\partial t}+TW-WT\right)\\ &=\frac{\partial\check{X}}{\partial w}T+\check{X}\frac{\partial W}{\partial t}+\frac{\partial\check{X}}{\partial t}W-\check{X}WT\\ &=\frac{\partial}{\partial t}(\check{X}W)+\left(\frac{\partial\check{X}}{\partial w}-\check{X}W\right)T.\end{aligned}$$

So, the function $Y_w(t):=\partial\check{X}/\partial w-\check{X}W$ satisfies the ordinary differential equation

$$\frac{dY_w}{dt}(t)=Y_w(t)T(t,w), \quad Y_w(0)=O$$

for each $w\in[0,1]$. Thus, by the uniqueness of the solution, $Y_w(t)=O$ holds on $[0,1]\times[0,1]$. Hence we have (2.9).

Thus, $\hat{X}(1)$ depends only on the end point P of the path. Hence we can set $X(\mathrm{P}):=\hat{X}(1)$ for each $\mathrm{P}\in U$, and obtain a map $X\colon U\to\mathrm{M}_n(\mathbb{R})$. Finally we show that X is the desired solution. The initial condition $X(\mathrm{P}_0)=X_0$ is obviously satisfied. On the other hand, if we set

$$Z(\delta):=X(u^1,\dots,u^j+\delta,\dots,u^m),$$

$Z(\delta)$ satisfies the equation (2.4) for the path $\gamma(\delta) := (u^1, \dots, u^j + \delta, \dots, u^m)$ with $Z(0) = X(\mathrm{P})$. Since $\Omega_\gamma = \Omega_j$,

$$\frac{\partial X}{\partial u^j}(\mathrm{P}) = \left.\frac{dZ}{d\delta}\right|_{\delta=0} = Z(0)\Omega_j(\mathrm{P}) = X(\mathrm{P})\Omega_j(\mathrm{P})$$

which completes the proof. □

Application: Poincaré's lemma.

Theorem 2.6 (Poincaré's lemma)**.** *If a differential* 1*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \dots, u^m)\, du^j$$

defined on a simply connected domain $U \subset \mathbb{R}^m$ is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on U such that $df = \omega$. Such a function f is unique up to additive constants.

Proof. Since

$$d\omega = \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} \right) du^i \wedge du^j,$$

the assumption is equivalent to

$$\frac{\partial \alpha_j}{\partial u^i} - \frac{\partial \alpha_i}{\partial u^j} = 0 \qquad (1 \leqq i < j \leqq m). \tag{2.11}$$

Consider a system of linear partial differential equations with unknown ξ, a 1×1-matrix valued function (i.e. a real-valued function), as

$$\frac{\partial \xi}{\partial u^j} = \xi \alpha_j \quad (j = 1, \dots, m), \qquad \xi(u_0^1, \dots, u_0^m) = 1. \tag{2.12}$$

Then it satisfies (2.3) because of (2.11). Hence by Theorem 4.5, there exists a smooth function $\xi(u^1, \dots, u^m)$ satisfying (2.12). In particular, Proposition 1.8 yields $\xi = \det \xi$ never vanishes. Hence $\xi(u_0^1, \dots, u_0^m) = 1 > 0$ means that $\xi > 0$ holds on U. Letting $f := \log \xi$, we have the function f satisfying $df = \omega$.

Next, we show the uniqueness: if two functions f and g satisfy $df = dg = \omega$, it holds that $d(f - g) = 0$. Hence by connectivity of U, $f - g$ must be constant. □

Application: Conjugation of Harmonic functions. In this paragraph, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. It is well-known that a smooth function

$$f \colon U \ni u + \mathrm{i}\, v \longmapsto \xi(u,v) + \mathrm{i}\, \eta(u,v) \in \mathbb{C} \qquad (\mathrm{i} = \sqrt{-1}) \tag{2.13}$$

defined on a domain $U \subset \mathbb{C}$ is *holomorphic* if and only if it satisfies the following relation, called the *Cauchy-Riemann equations*:

$$\frac{\partial \xi}{\partial u} = \frac{\partial \eta}{\partial v}, \qquad \frac{\partial \xi}{\partial v} = -\frac{\partial \eta}{\partial u}. \tag{2.14}$$

Definition 2.7. A function $f \colon U \to \mathbb{R}$ defined on a domain $U \subset \mathbb{R}^2$ is said to be *harmonic* if it satisfies

$$\Delta f = f_{uu} + f_{vv} = 0.$$

The operator Δ is called the *Laplacian*.

Proposition 2.8. *If function f in (2.13) is holomorphic, $\xi(u,v)$ and $\eta(u,v)$ are harmonic functions.*

Proof. By (2.14), we have

$$\xi_{uu} = (\xi_u)_u = (\eta_v)_u = \eta_{vu} = \eta_{uv} = (\eta_u)_v = (-\xi_v)_v = -\xi_{vv}.$$

Hence $\Delta\xi = 0$. Similarly,

$$\eta_{uu} = (-\xi_v)_u = -\xi_{vu} = -\xi_{uv} = -(\xi_u)_v = -(\eta_v)_v = -\eta_{vv}.$$

Thus $\Delta\eta = 0$. □

Theorem 2.9. *Let $U \subset \mathbb{C} = \mathbb{R}^2$ be a simply connected domain and $\xi(u,v)$ a C^∞-function harmonic on U*[6]*. Then there exists a C^∞ harmonic function η on U such that $\xi(u,v) + \mathrm{i}\,\eta(u,v)$ is holomorphic on U.*

Proof. Let $\alpha := -\xi_v\,du + \xi_u\,dv$. Then by the assumption,

$$d\alpha = (\xi_{vv} + \xi_{uu})\,du \wedge dv = 0$$

holds, that is, α is a closed 1-form. Hence by simple connectivity of U and the Poincaré's lemma (Theorem 4.8), there exists a function η such that $d\eta = \eta_u\,du + \eta_v\,dv = \alpha$. Such a function η satisfies (2.14) for given ξ. Hence $\xi + \mathrm{i}\,\eta$ is holomorphic in $u + \mathrm{i}\,v$. □

Example 2.10. A function $\xi(u,v) = e^u \cos v$ is harmonic. Set

$$\alpha := -\xi_v\,du + \xi_u\,dv = e^u \sin v\,du + e^u \cos v\,dv.$$

Then $\eta(u,v) = e^u \sin v$ satisfies $d\eta = \alpha$. Hence

$$\xi + \mathrm{i}\,\eta = e^u(\cos v + \mathrm{i} \sin v) = e^{u+\mathrm{i}\,v}$$

is holomorphic in $u + \mathrm{i}\,v$.

Definition 2.11. The harmonic function η in Theorem 2.9 is called the *conjugate* harmonic function of ξ.

Exercises

2-1 Let $\xi(u,v) := \log\sqrt{u^2+v^2}$ be a function defined on $U := \mathbb{R}^2 \setminus \{(0,0)\}$.

(1) Show that ξ is harmonic on U.

(2) Find the conjugate harmonic function η of ξ on

$$V = \mathbb{R}^2 \setminus \{(u,0) \mid u \leqq 0\} \subset U.$$

(3) Show that there exists no conjugate harmonic function of ξ defined on U.

2-2 Consider a linear system of partial differential equations for 3×3-matrix valued unknown X on a domain $U \subset \mathbb{R}^2$ as

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \qquad \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h^1_1 \\ \alpha & 0 & -h^2_1 \\ h^1_1 & h^2_1 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h^1_2 \\ \beta & 0 & -h^2_2 \\ h^1_2 & h^2_2 & 0 \end{pmatrix}\right),$$

where (u,v) are the canonical coordinate system of $\mathbb{R}^2$, and α, β and h^i_j $(i,j = 1,2)$ are smooth functions defined on U. Write down the integrability conditions in terms of α, β and h^i_j.

[6] The theorem holds under the assumption of C^2-differentiablity.

3 Differential Forms

Let M be an n-dimensional manifold and denote by $\mathcal{F}(M)$ and $\mathfrak{X}(M)$ the set of smooth function and the set of smooth vector fields on M, respectively.

Lie brackets A vector field $X \in \mathfrak{X}(M)$ can be considered as a differential operator acting on $\mathcal{F}(M)$ as $(Xf)(p) = X_p f$. By definition it satisfies the Leibniz rule

$$X(fg) = f(Xg) + g(Xf) \qquad (X \in \mathfrak{X}(M), f, g \in \mathcal{F}(M)). \tag{3.1}$$

For two vector fields X, $Y \in \mathfrak{X}(M)$, set

$$[X,Y]\colon \mathcal{F}(M) \ni f \longmapsto X(Yf) - Y(Xf) \in \mathcal{F}(M). \tag{3.2}$$

Then $[X,Y]$ also satisfies the Leibnitz rule (3.1), and gives a vector field on M. The map

$$[\ ,\ ]\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y) \mapsto [X,Y] \in \mathfrak{X}(M)$$

is called the *Lie bracket* on $\mathfrak{X}(M)$. One can easily show that the product $[\ ,\ ]$ is bilinear, skew symmetric and satisfies the *Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = \mathbf{0}, \tag{3.3}$$

that is, $(\mathfrak{X}(M), [\ ,\ ])$ is a *Lie algebra* (of infinite dimension). By the Leibniz rule, it holds that

$$[fX,Y] = f[X,Y] - (Yf)X, \quad [X,fY] = f[X,Y] + (Xf)Y \qquad (X,Y \in \mathfrak{X}(M), f \in \mathcal{F}(M)). \tag{3.4}$$

Tensors. For each $p \in M$, the *dual space* T_p^*M of T_pM is the liner space consisting of all linear maps from T_pM to $\mathbb{R}$.

Lemma 3.1. *Let $(x^1,\dots,x^n)$ be a local coordinate system of M around p, and set*

$$\left(\frac{\partial}{\partial x^j}\right)_p : \mathcal{F}(M) \ni f \mapsto \frac{\partial f}{\partial x^j}(p), \qquad (dx^j)_p\colon T_pM \to \mathbb{R} \qquad \textit{with} \qquad (dx^j)_p\left(\left(\frac{\partial}{\partial x^k}\right)_p\right) = \delta^j_k$$

for $j,k = 1,\dots,n$. Then $\{(\partial/\partial x^j)_p\}_{j=1,\dots,n}$ and $\{(dx^j)_p\}_{j=1,\dots,n}$ are a basis of T_pM and T_p^*M, respectively, where δ^j_k denotes Kronecker's delta symbol.

We let

$$T_p^*M \otimes T_p^*M \qquad \big(\text{resp.} \qquad T_p^*M \otimes T_p^*M \otimes T_p^*M\big)$$

the set of bilinear (resp. trilinear) maps of $T_pM \times T_pM$ (resp. $T_pM \times T_pM \times T_pM$) to $\mathbb{R}$. A section of the vector bundle

$$T^*M \otimes T^*M := \bigcup_{p\in M} T_p^*M \otimes T_p^*M \qquad \left(\text{resp. } T^*M \otimes T^*M \otimes T^*M := \bigcup_{p\in M} T_p^*M \otimes T_p^*M \otimes T_p^*M\right)$$

is called a *covariant* 2 (resp. 3)-*tensor*.

A section $\omega \in \Gamma(T^*M)$ of the cotangent bundle T^*M is called a *covariant* 1-*tensor* or a 1-*form*. A one form ω induces a linear map

$$\omega\colon \mathfrak{X}(M) \ni X \longmapsto \omega(X) \in \mathcal{F}(M), \qquad \text{where} \qquad \omega(X)(p) = \omega_p(X_p) \tag{3.5}$$

By definition, it holds that

$$\omega(fX) = f\omega(X) \qquad (f \in \mathcal{F}(M), X \in \mathfrak{X}(M)). \tag{3.6}$$

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Lemma 3.2. *A linear map* $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ *is a* 1*-form if and only if* (3.6) *holds.*

Proof. The "only if" part is trivial by definition. Assume a linear map $\omega\colon \mathfrak{X}(M) \to \mathcal{F}(M)$ satisfies (3.6). In fact, under a local coordinate system $(x^1, \dots, x^n)$ around $p \in M$,

$$\omega(X)(p) = \omega\left(\sum_{j=1}^n X^j \frac{\partial}{\partial x^j}\right)(p) = \sum_{j=1}^n X^j(p)\omega\left(\frac{\partial}{\partial x^j}\right)_p, \qquad \left(X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}.\right)$$

holds. In other words, $\omega(X)(p)$ depend only on X_p. Hence ω induces a map $\omega_p\colon T_pM \to \mathbb{R}$. $\square$

Similarly, a *covariant* 2 (resp. 3) tensor $\alpha \in \Gamma(T^*M \otimes T^*M)$ (resp. $\beta \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$)induces a bilinear (resp. trilinear) map $\alpha\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$. (resp. $\beta\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$. By the same reason as Lemma 3.2, we have

Lemma 3.3. *A bilinear map* $\alpha\colon \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathcal{F}(M)$ (*resp.* $\beta\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathcal{F}(M)$) *is a a covariant* 2 (*resp.* 3)*-tensor if and only if*

$$\alpha(fX,Y) = \alpha(X,fY) = f\alpha(X,Y)$$
$$\bigl(\textit{resp.}\quad \beta(fX,Y,Z) = \beta(X,fY,Z) = \beta(X,Y,fZ) = f\beta(X,Y,Z)\bigr)$$

holds for all X, Y, $Z \in \mathfrak{X}(M)$ *and* $f \in \mathcal{F}(M)$.

A covariant 2 (resp. 3)-tensor α (resp. β) said to be *skew-symmetric* if

$$\alpha(X,Y) = -\alpha(Y,X), \quad \bigl(\beta(X,Y,Z) = -\beta(Y,X,Z) = -\beta(X,Z,Y) = -\beta(Z,Y,X)\bigr)$$

holds for all X, Y, $Z \in \mathfrak{X}(M)$. We denote

$$\wedge^k(M) := \begin{cases} \mathcal{F}(M) & (k=0),\\ \Gamma(T^*M) & (k=1),\\ \bigl\{\omega \in \Gamma\bigl(T^*M \otimes T^*M\bigr)\,;\, \omega \text{ is skew-symmetric}\bigr\} & (k=2),\\ \bigl\{\omega \in \Gamma\bigl(T^*M \otimes T^*M \otimes T^*M\bigr)\,;\, \omega \text{ is skew-symmetric}\bigr\} & (k=3). \end{cases} \tag{3.7}$$

An element of $\wedge^k(M)$ is called an *k-form*.

The Exterior products. The *exterior product* $\alpha \wedge \beta \in \wedge^2(M)$ of two 1-forms α, $\beta \in \wedge^1(M)$ is defined as

$$(\alpha \wedge \beta)(X,Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X). \tag{3.8}$$

On the other hand, the exterior product of α and ω is defined as a 3-form on M by

$$(\alpha \wedge \omega)(X,Y,Z) = (\omega \wedge \alpha)(X,Y,Z) := \alpha(X,Y)\omega(Z) + \alpha(Y,Z)\omega(X) + \alpha(Z,X)\omega(Y). \tag{3.9}$$

Then by a direct computation together with (3.8), it holds that

$$(\mu \wedge \omega) \wedge \lambda = \mu \wedge (\omega \wedge \lambda)\Bigl(=: \mu \wedge \omega \wedge \lambda\Bigr) \tag{3.10}$$

for 1-forms μ, ω and λ.

The Exterior derivative. Under a local coordinate system $(x^1, \dots, x^n)$, a one form α and a two form ω are expressed as

$$\alpha = \sum_{j=1}^{n} \alpha_j \, dx^j, \qquad \omega = \sum_{1 \leqq i < j \leqq n} \omega_{ij} \, dx^i \wedge dx^j,$$

where α_j $(j = 1, \dots, n)$ and ω_{ij} $(1 \leqq i < j \leqq n)$ are smooth functions in $(x^1, \dots, x^n)$. By Lemma 3.3 and the property (3.4) of the Lie brackets, we have

Lemma 3.4. *For a function $f \in \mathcal{F}(M) = \wedge^0(M)$, a 1-form $\alpha \in \wedge^1(M)$ and a 2-form $\beta \in \wedge^2(M)$)*

$$\begin{aligned}
df &\colon \mathfrak{X}(M) \ni X \mapsto df(X) = Xf \in \mathcal{F}(M),\\
d\alpha &\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]) \in \mathcal{F}(M)\\
d\beta &\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y, Z) \mapsto\\
&\quad X\beta(Y, Z) + Y\beta(Z, X) + Z\beta(X, Y) - \beta([X, Y], Z) - \beta([Y, Z], Z) - \beta([Z, X], Y)
\end{aligned}$$

are a 1-form, a 2-form and a 3-form respectively.

Definition 3.5. For a function f, a 1-form α and a 2-form β, df, $d\alpha$ and $d\beta$ are called the *exterior derivatives* of f, α and β, respectively.

Then, for one forms μ and ω, we have

$$dd\omega = 0, \qquad d(\mu \wedge \omega) = d\mu \wedge \omega - \mu \wedge d\omega, \tag{3.11}$$

by the definition and the Jacobi identity (3.3).

The Riemannian connection. In the rest of this section, we let (M, g) be an n-dimensional (pseudo) Riemannian manifold, and denote by $\langle\ ,\ \rangle$ the inner product induced by g.

Lemma 3.6. *There exists the unique bilinear map $\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto \nabla_X Y \in \mathfrak{X}(M)$ satisfying*

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle X, \nabla_X Z\rangle \qquad (X, Y, Z \in \mathfrak{X}(M)) \tag{3.12}$$

Definition 3.7. The map ∇ in Lemma 3.6 is called the *Riemannian connection* or the *Levi-Civita connection* of (M, g).

Lemma 3.8. *The Riemannian connection ∇ satisfies*

$$\nabla_{fX} Y = f \nabla_X Y, \qquad \nabla_X (fY) = (Xf)Y + f\nabla_X Y. \tag{3.13}$$

Remark 3.9. A bilinear map $\nabla \colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying (3.13) is called a *linear connection* or an *affine connection*.

Remark 3.10. By Lemmas 3.8 and 3.2, $X \mapsto \nabla_X Y$ determines a one form.

Orthonormal frames. For a sake of simplicity, we assume that g is positive definite, in other words, (M, g) is a Riemannian manifold.

Definition 3.11. Let $U \subset M$ be a domain of M. An n-tuple of vector fields $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ on U is called an *orthonormal frame* on U if $\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \delta_{ij}$. It is said to be *positive* if M is oriented and $\{\boldsymbol{e}_j\}$ is compatible to the orientation on M.

Remark 3.12. For each $p \in M$, there exists a neighborhood U of p which admits an orthonormal frame on U.

Lemma 3.13. *Let* $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *be two orthonormal frames on* $U \subset M$*. Then there exists a smooth map*

$$\Theta\colon U \longrightarrow \mathrm{O}(n) \qquad \textit{such that} \qquad [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n] = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]\Theta. \tag{3.14}$$

Moreover, if $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *determines the common orientation,* Θ *is valued on* $\mathrm{SO}(n)$*.*

The map Θ in Lemma 3.13 is called a *gauge transformation*.

For an orthonormal frame $\{\boldsymbol{e}_j\}$ on U, we denote by $\{\omega^j\}_{j=1,\dots,n}$ the *dual frame* of $\{\boldsymbol{e}_j\}$, that is, $\omega^j \in \wedge^1(U)$ such that

$$\omega^j(\boldsymbol{e}_k) = \delta^j_k = \begin{cases} 1 & (j=k) \\ 0 & (\text{otherwise}). \end{cases}$$

In other words, $\omega^j(X) = \langle \boldsymbol{e}_j, X\rangle$.

Lemma 3.14. *Two orthonormal frames* $\{\boldsymbol{e}_j\}$ *and* $\{\boldsymbol{v}_j\}$ *are related as* (3.14)*. Then their duals* $\{\omega^j\}$ *and* $\{\lambda^j\}$ *satisfy*

$$\begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix} = \Theta \begin{pmatrix}\omega^1\\ \vdots \\ \omega^n\end{pmatrix}.$$

Proof.

$$\begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n) = \begin{pmatrix}\lambda^1\\ \vdots \\ \lambda^n\end{pmatrix}(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n)\Theta = \Theta = \Theta\begin{pmatrix}\omega^1\\ \vdots \\ \omega^n\end{pmatrix}(\boldsymbol{e}_1,\dots,\boldsymbol{e}_n). \qquad \square$$

Connection forms.

Definition 3.15. The *connection form* with respect to an orthonormal frame $\{\boldsymbol{e}_j\}$ is a $n\times n$-matrix valued one form Ω on U defined by

$$\Omega = \begin{pmatrix} \omega^1_1 & \omega^1_2 & \dots & \omega^1_n \\ \omega^2_1 & \omega^2_2 & \dots & \omega^2_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega^n_1 & \omega^n_2 & \dots & \omega^n_n \end{pmatrix}, \qquad \omega^k_j := \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle \in \wedge^1(U).$$

By definition, we have $\nabla \boldsymbol{e}_j = \sum_{k=1}^n \omega^k_j \boldsymbol{e}_k$, that is, $\nabla[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n] = [\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]\Omega$.

Lemma 3.16. $\omega^k_j = -\omega^j_k$.

Proof. $\omega^k_j = \langle \nabla \boldsymbol{e}_j, \boldsymbol{e}_k\rangle = d\,\langle \boldsymbol{e}_j, \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_j, \nabla\boldsymbol{e}_k\rangle = -\omega^j_k$.

$\square$

Lemma 3.17. $d\omega^i = \sum_{l=1}^n \omega^l \wedge \omega^i_l$.

Proof.

$$\begin{aligned} d\omega^i(\boldsymbol{e}_j,\boldsymbol{e}_k) &= \boldsymbol{e}_j\omega^i(\boldsymbol{e}_k) - \boldsymbol{e}_k\omega^i(\boldsymbol{e}_j) - \omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k]) = -\omega^i([\boldsymbol{e}_j,\boldsymbol{e}_k]) \\ &= -\omega^i(\nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j) = -\left\langle \nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j, \boldsymbol{e}_i\right\rangle = -\omega^i_k(\boldsymbol{e}_j) + \omega^i_j(\boldsymbol{e}_k) \\ &= \sum_{l=1}^n \bigl(-\omega^i_l(\boldsymbol{e}_j)\omega^l(\boldsymbol{e}_k) + \omega^i_l(\boldsymbol{e}_k)\omega^l(\boldsymbol{e}_j)\bigr) = \sum_{l=1}^n \omega^l \wedge \omega^i_l(\boldsymbol{e}_j,\boldsymbol{e}_k). \qquad \square \end{aligned}$$

Exercises

3-1 Let $\{\boldsymbol{e}_j\}$ and $\{\boldsymbol{v}_j\}$ be two orthonormal frames on a domain U of a Riemannian n-manifold M, which are related as (3.14). Show that the connection forms Ω of $\{\boldsymbol{e}_j\}$ and Λ of $\{\boldsymbol{v}_j\}$ satisfy $\Omega = \Theta^{-1}\Lambda\Theta + \Theta^{-1}d\Theta$.

3-2 Let $\mathbb{R}^3_1$ be the 3-dimensional Lorentz-Minkowski space and let $H^2(-1)$ the hyperbolic 2-space (i.e. the hyperbolic plane) of constant curvature -1.

(1) Verify that

$$\boldsymbol{f}(u,v) := (\cosh u, \cos v \sinh u, \sin v \sinh u)$$

gives a local coordinate system on $U := H^2(-1) \setminus \{(1,0,0)\}$, and

$$\boldsymbol{e}_1 := (\sinh u, \cos v \cosh u, \sin v \cosh u), \qquad \boldsymbol{e}_2 := (0, -\sin v, \cos v)$$

forms a orthonormal frame on U.

(2) Compute the connection form(s) with respect to the orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$.

4 Curvatre forms

4.1 Addendum to the previous section

Proposition 4.1 (The local expression of the Lie bracket)**.** *Let $(U; x^1, \dots, x^n)$ be a coordinate neighborhood of an n-manifold M. Then the Lie bracket of two vector fields*

$$X = \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial x^j}, \qquad Y = \sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j}$$

is expressed as

$$[X, Y] = \sum_{j=1}^{n} \left(\xi^k \frac{\partial \eta^j}{\partial x^k} - \eta^k \frac{\partial \xi^j}{\partial x^k} \right) \frac{\partial}{\partial x^j}.$$

Proof. For a smooth function f on U, it holds that

$$\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f.$$

Hence $[\partial/\partial x^i, \partial/\partial x^j] = 0$. Then the conclusion follows from bilinearlity of $[X, Y]$ and the formula

$$[fX, Y] = f[X, Y] - (Yf)X, \qquad [X, fY] = f[X, Y] + (Xf)Y$$

for a smooth function f and vector fields X and Y. □

Proposition 4.2 (A local expression of the connection forms)**.** *Let U be a domain of a Riemannian n-manifold (M, g) and $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ an orthonormal frame on U. Then the connection form ω_i^j with respect to the frame $[\boldsymbol{e}_j]$ is obtained as*

$$\omega_i^j(\boldsymbol{e}_k) = \frac{1}{2}\Big(-\langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle + \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i\rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j\rangle \Big),$$

where $\langle\ ,\ \rangle$ denotes the inner product induced from g.

Proof. By the definition of the Levi-Civita connection ∇,

$$\begin{aligned}
\omega_i^j(\boldsymbol{e}_k) &= \langle \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_i, \boldsymbol{e}_j\rangle = \boldsymbol{e}_k \langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle - \langle \boldsymbol{e}_i, \nabla_{\boldsymbol{e}_k}\boldsymbol{e}_j\rangle = -\left\langle \boldsymbol{e}_i, \nabla_{\boldsymbol{e}_j}\boldsymbol{e}_k + [\boldsymbol{e}_k, \boldsymbol{e}_j]\right\rangle \\
&= -\boldsymbol{e}_j \langle \boldsymbol{e}_i, \boldsymbol{e}_k\rangle + \left\langle \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_i, \boldsymbol{e}_k\right\rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k]\rangle \\
&= \langle \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j, \boldsymbol{e}_k\rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k]\rangle \\
&= \boldsymbol{e}_i \langle \boldsymbol{e}_j, \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_j, \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_k\rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k]\rangle \\
&= -\langle \boldsymbol{e}_j, \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_i\rangle - \langle \boldsymbol{e}_j, [\boldsymbol{e}_i, \boldsymbol{e}_k]\rangle + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle - \langle \boldsymbol{e}_i, [\boldsymbol{e}_j, \boldsymbol{e}_k]\rangle \\
&= -\omega_i^j(\boldsymbol{e}_k) + \langle [\boldsymbol{e}_i, \boldsymbol{e}_j], \boldsymbol{e}_k\rangle - \langle [\boldsymbol{e}_j, \boldsymbol{e}_k], \boldsymbol{e}_i\rangle + \langle [\boldsymbol{e}_k, \boldsymbol{e}_i], \boldsymbol{e}_j\rangle .
\end{aligned}$$ □

4.2 Preliminaries

Integrability condition, a review. Let U be a domain of $\mathbb{R}^m$ with coordinate system $(x^1, \dots, x^m)$, and consider a system of differential equations

$$\frac{\partial F}{\partial x^l} = F\Omega_l \qquad (l = 1, \dots, m) \tag{4.1}$$

with initial condition

$$F(\mathrm{P}_0) = F_0 \in \mathrm{M}_n(\mathbb{R}), \qquad \mathrm{P}_0 = (x_0^1, \dots, x_0^m) \in U, \tag{4.2}$$

where F is an unknown map into the space of $n \times n$-real matrices $\mathrm{M}_n(\mathbb{R})$, and the coefficient matrices Ω_l $(l = 1, \dots, m)$ are $\mathrm{M}_n(\mathbb{R})$-valued C^∞-functions.

Lemma 4.3. *If the initial condition F_0 in* (4.2) *is non-singular, i.e., $F_0 \in \mathrm{GL}(n, \mathbb{R})$*[7]*, F satisfying*

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[7] $\mathrm{GL}(n, \mathbb{R})$ denotes the set of $n \times n$-regular matrices.

(4.1) *is a* $\mathrm{GL}(n,\mathbb{R})$*-valued function, that is,* F *is invertible for each point on* U*.*

Proof. For each $\mathrm{P} \in U$, take a smooth path $\gamma(t) := (x^1(t), \dots, x^m(t))$ $(0 \leqq t \leqq 1)$ with $\gamma(0) = \mathrm{P}_0$ and $\gamma(1) = \mathrm{P}$. Then the matrix-valued function $\hat{F} := F \circ \gamma$ of one variable satisfies the ordinary differential equation

$$\frac{d\hat{F}}{dt} = \hat{F}\hat{\Omega}, \qquad \hat{\Omega} := \sum_{l=1}^{m} \Omega_l \circ \gamma \frac{dx^l}{dt}.$$

Hence $\varphi := \det \hat{F}$ satisfies

$$\frac{d\varphi}{dt} = \frac{d}{dt}\det\hat{F} = \operatorname{tr}\left(\widetilde{\hat{F}}\frac{d\hat{F}}{dt}\right) = \operatorname{tr}(\widetilde{\hat{F}}\hat{F}\hat{\Omega}) = \det\hat{F}\operatorname{tr}\hat{\Omega} = \varphi\omega$$

where $\widetilde{\hat{F}}$ denotes the cofactor matrix of $\hat{F}$ and $\omega := \operatorname{tr}\hat{\Omega}$. So

$$\det\hat{F}(t) = \varphi(t) = \varphi_0 \exp\int_0^t \omega(\tau)\,d\tau \qquad (\varphi_0 := \det F_0),$$

proving the lemma. □

As seen in the previous lectures the following *integrability condition* holds:

Lemma 4.4. *If a* C^∞*-map* $F\colon U \to \mathrm{GL}(n,\mathbb{R})$ *satisfies* (4.1)*, then it hold on* U *that*

$$\frac{\partial\Omega_l}{\partial x^k} - \frac{\partial\Omega_k}{\partial x^l} + \Omega_k\Omega_l - \Omega_l\Omega_k = O \qquad (1 \leqq k < l \leqq m). \tag{4.3}$$

The integrability condition (4.3) guarantees existence of the solution of (4.1) as follows

Theorem 4.5. *Let* $\Omega_l\colon U \to \mathrm{M}_m(\mathbb{R})$ $(l = 1, \dots, n)$ *be* C^∞*-functions defined on a* <u>*simply connected*</u> *domain* $U \subset \mathbb{R}^n$ *satisfying* (4.3) *Then for each* $\mathrm{P}_0 \in U$ *and* $F_0 \in \mathrm{M}_m(\mathbb{R})$*, there exists the unique* $m \times m$*-matrix valued function* $F\colon U \to \mathrm{M}_m(\mathbb{R})$ *satisfying* (4.1) *and* (4.2)*. Moreover,*

- *if* $F_0 \in \mathrm{GL}(m,\mathbb{R})$*,* $F(\mathrm{P}) \in \mathrm{GL}(m,\mathbb{R})$ *holds on* U*,*
- *if* $F_0 \in \mathrm{SO}(m)$ *and* Ω_l*'s are skew-symmetric matrices,* $F(\mathrm{P}) \in \mathrm{SO}(m)$ *holds on* U*.*

Coordinate-free expressions Let $\Omega_l\colon U \to \mathrm{M}_n(\mathbb{R})$ $(l = 1, \dots, m)$ be C^∞-functions defined on a domain $U \subset \mathbb{R}^m$, and define $n \times n$-matrix Ω of 1-forms as

$$\Omega = \begin{pmatrix} \omega_1^1 & \omega_2^1 & \dots & \omega_n^1 \\ \omega_1^2 & \omega_2^2 & \dots & \omega_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^n & \omega_2^n & \dots & \omega_n^n \end{pmatrix} := \sum_{l=1}^{m} \Omega_l\,dx^l = \begin{pmatrix} \sum\omega_{l,1}^1\,dx^l & \sum\omega_{l,2}^1\,dx^l & \dots & \sum\omega_{l,n}^1\,dx^l \\ \sum\omega_{l,1}^2\,dx^l & \sum\omega_{l,2}^2\,dx^l & \dots & \sum\omega_{l,n}^2\,dx^l \\ \vdots & \vdots & \ddots & \vdots \\ \sum\omega_{l,1}^n\,dx^l & \sum\omega_{l,2}^n\,dx^l & \dots & \sum\omega_{l,n}^n\,dx^l \end{pmatrix}, \tag{4.4}$$

where $\Omega_l = (\omega_{l,j}^i)$. Then Ω is considered as a $\mathrm{M}_n(\mathbb{R})$-valued 1-form, and (4.1) is restated as

$$dF = F\Omega. \tag{4.5}$$

Lemma 4.6. *Under the situation above, the integrability condition* (4.3) *is equivalent to*

$$d\Omega + \Omega\wedge\Omega = O, \qquad \text{where} \quad \Omega\wedge\Omega = \left(\sum_{k=1}^{n}\omega_k^i\wedge\omega_j^k\right)_{i,j=1,\dots,n}. \tag{4.6}$$

Proof. Assume F be a solution of (4.5) with $F \in \mathrm{GL}(n, \mathbb{R})$. Then

$$O = ddF = d(F\Omega) = dF \wedge \Omega + F\,d\Omega = F(\Omega \wedge \Omega + d\Omega). \qquad \square$$

Thus, by using differential forms, we can state the system of partial differential equations (4.1) and its integrability condition (4.3) in coordinate-free form. The proof of Theorem 4.5 works not only simply connected domain $U \subset \mathbb{R}^m$ but also simply connected m-manifold, and thus, we have

Theorem 4.7. *Let Ω be an $\mathrm{M}_n(\mathbb{R})$-valued 1-form on a simply connected m-manifold M satisfying (4.6). Then for each $\mathrm{P}_0 \in M$ and $F_0 \in \mathrm{M}_n(\mathbb{R})$, there exists the unique $n \times n$-matrix valued function $F \colon M \to \mathrm{M}_n(\mathbb{R})$ satisfying (4.5) with $F(\mathrm{P}) = F_0$. Moreover,*

- *if $F_0 \in \mathrm{GL}(n, \mathbb{R})$, $F(\mathrm{P}) \in \mathrm{GL}(n, \mathbb{R})$ holds on M,*
- *if $F_0 \in \mathrm{SO}(n)$ and Ω is skew-symmetric, $F(\mathrm{P}) \in \mathrm{SO}(n)$ holds on M.*

When $n = 1$, that is, Ω is a usual 1-form, $\Omega \wedge \Omega$ always vanishes, and the integrability condition (4.6) is simply $d\Omega = 0$. Then we have the following Poncaré's lemma[8].

Theorem 4.8 (Poincaré's lemma)**.** *If a differential 1-form ω defined on a simply connected and connected m-manifold M is closed, that is, $d\omega = 0$ holds, then there exists a C^∞-function f on M such that $df = \omega$. Such a function f is unique up to additive constants.*

Proof. Since ω is closed, there exists a function F on M satisfying $dF = F\omega$ with initial condition $F(\mathrm{P}_0) = 1$. By Lemma 4.3, F does not vanish on M, that is, $F > 0$. Hence $f := \log F$ is a smooth function on M satisfying $df = dF/F = F\omega/F = \omega$. Take another function g on M satisfying $dg = \omega$, $d(f - g) = 0$ holds. Then connectedness of M infers that $f - g$ is constant. $\square$

4.3 Curvature form

Let U be a domain of n-dimensional Riemannian manifold (M, g). We let Ω be the connection form with respect to an orthonormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on U, as defined in Definition 3.15.

Definition 4.9. We define a skew-symmetric matrix-valued 2-form by $K := d\Omega + \Omega \wedge \Omega$ and call the *curvature form* with respect to the frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$.

Take an orthonormal frame $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ on U and take a gauge transformation $\Theta \colon U \to \mathrm{O}(n)$:

$$[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta.$$

Denoting the connection form and the curvature form with respect to $[\boldsymbol{v}_j]$ by $\widetilde{\Omega}$ and $\widetilde{K}$. Then

Proposition 4.10. (1) $\Omega = \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta$, (2) $K = \Theta^{-1}\widetilde{K}\Theta$.

Proof. Since

$$\begin{aligned}
[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Omega &= \nabla[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = \nabla([\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta) = \nabla[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta + [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]d\Theta \\
&= [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\widetilde{\Omega}\Theta + [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]d\Theta = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Theta^{-1}(\widetilde{\Omega}\Theta + d\Theta),
\end{aligned}$$

the first assertion is obtained. Next, noticing $d(\widetilde{\Omega}\Theta) = (d\widetilde{\Omega})\Theta - \widetilde{\Omega} \wedge d\Theta$, $\widetilde{\Omega}\Theta^{-1} \wedge \Theta\widetilde{\Omega} = \widetilde{\Omega} \wedge \widetilde{\Omega}$, and so on, we have

$$\begin{aligned}
d\Omega + \Omega \wedge \Omega &= d(\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) + (\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) \wedge (\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta) \\
&= -\Theta^{-1}d\Theta\Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\widetilde{\Omega}\Theta - \Theta^{-1}\widetilde{\Omega} \wedge d\Theta - \Theta^{-1}d\Theta\Theta^{-1} \wedge d\Theta \\
&\quad + \Theta^{-1}\widetilde{\Omega}\Theta \wedge \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}\widetilde{\Omega}\Theta + \Theta^{-1}\widetilde{\Omega}\Theta \wedge \Theta^{-1}d\Theta + \Theta^{-1}d\Theta \wedge \Theta^{-1}d\Theta \\
&= \Theta^{-1}(d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega})\Theta,
\end{aligned}$$

proving (2). $\square$

[8]Theorem 2.6 in Advanced Topics in Geometry E (MTH.B501).

The goal of this section is to prove the following

Theorem 4.11. *Let U be a domain of a Riemannian n-manifold (M, g) and K the curvature form with respect to an orthonormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on U. For a point $\mathrm{P} \in U$, there exists a local coordinate system $(x^1, \dots, x^n)$ around P such that $[\partial/\partial x^1, \dots, \partial/\partial x^n]$ is an orthonormal frame if and only if K vanishes on a neighborhood of P.*

Remark 4.12. By (2) of Proposition 4.10, the condition $K = 0$ does not depend on choice of orthonormal frames. A Riemannian manifold (M, g) said to be *flat* if $K = 0$ holds on M.

Proof of Theorem 4.11. First, we shall show the "only if" part: Let $(x^1, \dots, x^n)$ be a coordinate system such that $[\boldsymbol{e}_j := \partial/\partial x^j]$ is an orthonormal frame. Since

$$[\boldsymbol{e}_j, \boldsymbol{e}_k] = \left[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right] = \mathbf{0},$$

Proposition 4.2 yields that all components of the connection forms ω_i^j vanish. Hene we have $K = 0$.

Conversely, assume $K = 0$ for an orthonormal frame $[\boldsymbol{e}_j]$. Since the connection form Ω satisfies $d\Omega + \Omega \wedge \Omega = O$, there exists a matrix-valued function $\Theta\colon V \to \mathrm{SO}(n)$ satisfying $d\Theta = \Theta\Omega$, $\Theta(\mathrm{P}) = \mathrm{id}$ on a sufficiently small neighborhood V of P, because of Theorem 4.5. Take a new orthonormal frame $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] := [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Theta^{-1}$. Then by (1) of Proposition 4.10, the connection form $\widetilde{\Omega} = (\tilde{\omega}_i^j)$ with respect to $[\boldsymbol{v}_j]$ vanishes identically. So by Lemma 3.17, $d\omega^i = 0$ holds for $i = 1, \dots, n$. Hence by the Poincaré Lemma (Theorem 4.8), there exists a smooth functions on a neighborhood V of P. Such $(x^1, \dots, x^n)$ is a desired coordinate system if V is sufficiently small. □

Exercises

4-1 Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\, d\theta^2 \qquad \text{on} \qquad U := \{(r, \theta)\,;\, 0 < r < r_0, -\pi < \theta < \pi\},$$

where $r_0 \in (0, +\infty]$ and φ is a positive smooth function defined on $(0, r_0)$ with

$$\lim_{r \to +0} \varphi(r) = 0, \qquad \lim_{r \to +0} \varphi'(r) = 1.$$

Find a function φ such that (U, g) is flat. (Hint: $[\partial/\partial r, (1/\varphi)\partial/\partial\theta)]$ is an orthonormal frame.)

4-2 Compute the curvature form of $H^2(-1)$ with respect to an orthonormal frame $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ as in Exercise 3-2.

5 The Sectional Curvature

5.1 Preliminaries

Exterior products of tangent vectors. Let V be an n-dimensional vector space ($1 \leqq n < \infty$) and denote by V^* its dual. Then $(V^*)^*$ can be naturally identified with V itself. In fact,

$$I : V \ni \boldsymbol{v} \longmapsto I_{\boldsymbol{v}} \in (V^*)^* := \{A\colon V^* \to \mathbb{R}; \text{linear}\}, \qquad I_{\boldsymbol{v}}(\alpha) := \alpha(\boldsymbol{v})$$

is a linear map with trivial kernel. Then I is an isomorphism because $\dim (V^*)^* = \dim V$.

We denote by $\wedge^2 V := \wedge^2 (V^*)^*$ the set of skew-symmetric bilinear forms on V^*. For vectors $\boldsymbol{v}$, $\boldsymbol{w} \in V$, the *exterior product* of them is an element of $\wedge^2 V$ defined as

$$(\boldsymbol{v} \wedge \boldsymbol{w})(\alpha, \beta) := \alpha(\boldsymbol{v})\beta(\boldsymbol{w}) - \alpha(\boldsymbol{w})\beta(\boldsymbol{v}) \qquad (\alpha, \beta \in V^*).$$

For a basis $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ on V,

$$\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j; 1 \leqq i < j \leqq n\} \tag{5.1}$$

is a basis of $\wedge^2 V$. In particular $\dim \wedge^2 V = \frac{1}{2}n(n-1)$. When V is a vector space endowed with an inner product $\langle\ ,\ \rangle$ and $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ is an orthonormal basis, there exists the unique inner product, which is also denoted by $\langle\ ,\ \rangle$, of $\wedge^2 V$ such that (5.1) is an orthonormal basis. This definition of the inner product does not depend on choice of orthonormal bases of V. In fact, take another orthonormal basis $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ related with $[\boldsymbol{e}_j]$ by

$$[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]\Theta \qquad \Theta = (\theta^j_i) \in \mathrm{O}(n).$$

Since $\Theta^T = \Theta^{-1}$, $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] = [\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]\Theta^T$ holds. Hence

$$\boldsymbol{v}_s \wedge \boldsymbol{v}_t = \left(\sum_i \theta^i_s \boldsymbol{e}_i\right) \wedge \left(\sum_j \theta^j_t \boldsymbol{e}_j\right) = \sum_{i,j} \theta^s_i \theta^t_j (\boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \sum_{i<j} \left(\theta^s_i \theta^t_j - \theta^s_j \theta^t_i\right)(\boldsymbol{e}_i \wedge \boldsymbol{e}_j),$$

and so

$$\begin{aligned}
\langle \boldsymbol{v}_s \wedge \boldsymbol{v}_t, \boldsymbol{v}_u \wedge \boldsymbol{v}_v \rangle &= \sum_{i<j,k<l} (\theta^s_i \theta^t_j - \theta^s_j \theta^t_i)(\theta^u_k \theta^v_l - \theta^u_l \theta^v_k) \langle \boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_k \wedge \boldsymbol{e}_l \rangle \\
&= \sum_{i<j,k<l} (\theta^s_i \theta^t_j - \theta^s_j \theta^t_i)(\theta^u_k \theta^v_l - \theta^u_l \theta^v_k)\delta_{ik}\delta_{jl} = \sum_{i<j} (\theta^s_i \theta^t_j - \theta^s_j \theta^t_i)(\theta^u_i \theta^v_j - \theta^u_j \theta^v_i) \\
&= \sum_{i<j} (\theta^s_i \theta^t_j \theta^u_i \theta^v_j - \theta^s_j \theta^t_i \theta^u_i \theta^v_j - \theta^s_i \theta^t_j \theta^u_j \theta^v_i + \theta^s_j \theta^t_i \theta^u_j \theta^v_i) \\
&= \sum_{i<j} \theta^s_i \theta^t_j \theta^u_i \theta^v_j + \sum_{i<j} \theta^s_j \theta^t_i \theta^u_i \theta^v_j - \sum_{i>j} \theta^s_j \theta^t_i \theta^u_i \theta^v_j + \sum_{i>j} \theta^s_i \theta^t_j \theta^u_i \theta^v_j \\
&= \sum_{i \neq j} \theta^s_i \theta^t_j \theta^u_i \theta^v_j - \sum_{i \neq j} \theta^s_j \theta^t_i \theta^u_i \theta^v_j \\
&= \sum_{i,j} (\theta^s_i \theta^t_j \theta^u_i \theta^v_j - \theta^s_j \theta^t_i \theta^u_i \theta^v_j) - \sum_i (\theta^s_i \theta^t_i \theta^u_i \theta^v_i - \theta^s_i \theta^t_i \theta^u_i \theta^v_i) \\
&= \delta^{su}\delta^{tv} - \delta^{tu}\delta^{sv}
\end{aligned}$$

because $\sum_i \theta^s_i \theta^t_i = \delta^{st}$. So, if $s < t$ and $u < v$, the second term of the right-hand side vanishes. That is, $\{\boldsymbol{v}_s \wedge \boldsymbol{v}_t\,;\, s < t\}$ is an orthonormal basis as well as $\{\boldsymbol{e}_i \wedge \boldsymbol{e}_j\,;\, i < j\}$ is.

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Symmetric bilinear forms. Let V be a real vector space. A bilinear map $q\colon V\times V\to\mathbb{R}$ is said to be *symmetric* if $q(\boldsymbol{v},\boldsymbol{w})=q(\boldsymbol{w},\boldsymbol{v})$ for all $\boldsymbol{v},\ \boldsymbol{w}\in V$.

Lemma 5.1. *Two symmetric bilinear forms q and q' coincide with each other if and only if $q(\boldsymbol{v},\boldsymbol{v})=q'(\boldsymbol{v},\boldsymbol{v})$ hold for all $\boldsymbol{v}\in V$.*

Proof. By symmetricity, $q(\boldsymbol{v},\boldsymbol{w})=\frac{1}{2}(q(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w})-q(\boldsymbol{v},\boldsymbol{v})-q(\boldsymbol{w},\boldsymbol{w}))$ holds. □

5.2 Sectional Curvature

Let U be a domain on a Riemannian n-manifold (M,g), and $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ an orthonormal frame on U. Denote by $(\omega^j)_{j=1,\dots,n}$, $\varOmega=(\omega_i^j)_{i,j=1,\dots,n}$ and $K=(\kappa_i^j)_{i=1,\dots,n}:=d\varOmega+\varOmega\wedge\varOmega$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$. Then Lemma 3.17 and Definition 4.9, we have

$$d\omega^j=\sum_l \omega^l\wedge\omega_l^j,\qquad \kappa_i^j=d\omega_i^j+\sum_l\omega_l^j\wedge\omega_i^l. \tag{5.2}$$

Since $\varOmega$ is a one form valued in the skew-symmetric matrices, so is K:

$$\omega_i^j=-\omega_j^i,\qquad \kappa_i^j=-\kappa_j^i. \tag{5.3}$$

Proposition 5.2 (The first Bianchi identity)**.** $\kappa_j^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa_k^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa_l^i(\boldsymbol{e}_j,\boldsymbol{e}_k)=0$.

Proof. By (5.2) and (3.11),

$$\begin{aligned}
0=dd\omega^i&=d\left(\sum_s\omega^s\wedge\omega_s^i\right)=\sum_s\left(d\omega^s\wedge\omega_s^i-\omega^s\wedge\omega_s^i\right)\\
&=\sum_s\left(\sum_m(\omega^m\wedge\omega_m^s)\wedge\omega_s^i-\omega^s\wedge\left(\kappa_s^i-\sum_m\omega_m^i\wedge d\omega_s^m\right)\right)\\
&=\sum_{s,m}\omega^m\wedge\omega_m^s\wedge\omega_s^i+\sum_{s,m}\omega^s\wedge\omega_m^i\wedge\omega_s^m-\sum_s\omega^s\wedge\kappa_s^i\\
&=\sum_{s,m}\omega^m\wedge(\omega_m^s\wedge\omega_s^i+\omega_s^i\wedge\omega_m^s)-\sum_s\omega^s\wedge\kappa_s^i=-\sum_s\omega^s\wedge\kappa_s^i.
\end{aligned}$$

Hence

$$\begin{aligned}
0&=\sum_s(\omega^s\wedge\kappa_s^i)(\boldsymbol{e}_j,\boldsymbol{e}_k,\boldsymbol{e}_l)=\sum_s\big(\omega^s(\boldsymbol{e}_j)\kappa_s^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\omega^s(\boldsymbol{e}_k)\kappa_s^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\omega^s(\boldsymbol{e}_l)\kappa_s^i(\boldsymbol{e}_j,\boldsymbol{e}_k)\big)\\
&=\sum_s\big(\delta_j^s\kappa_s^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\delta_k^s\kappa_s^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\delta_l^s\kappa_s^i(\boldsymbol{e}_j,\boldsymbol{e}_k)\big)\\
&=\kappa_j^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa_k^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa_l^i(\boldsymbol{e}_j,\boldsymbol{e}_k),
\end{aligned}$$

proving the assertion. □

Corollary 5.3. $\kappa_j^i(\boldsymbol{e}_k,\boldsymbol{e}_l)=\kappa_l^k(\boldsymbol{e}_i,\boldsymbol{e}_j)$.

Proof. By Proposition 5.2,

$$\begin{aligned}
\kappa_j^i(\boldsymbol{e}_k,\boldsymbol{e}_l)+\kappa_k^i(\boldsymbol{e}_l,\boldsymbol{e}_j)+\kappa_l^i(\boldsymbol{e}_j,\boldsymbol{e}_k)&=0\\
\kappa_k^j(\boldsymbol{e}_i,\boldsymbol{e}_l)+\kappa_i^j(\boldsymbol{e}_l,\boldsymbol{e}_k)+\kappa_l^j(\boldsymbol{e}_k,\boldsymbol{e}_i)&=0\\
\kappa_i^k(\boldsymbol{e}_j,\boldsymbol{e}_l)+\kappa_j^k(\boldsymbol{e}_l,\boldsymbol{e}_i)+\kappa_l^k(\boldsymbol{e}_i,\boldsymbol{e}_j)&=0.
\end{aligned}$$

Summing up these and noticing $\kappa_i^j=-\kappa_j^i$, we have the conclusion. □

A quadratic form induced from the curvature form. We fix a point $p \in U$. Under the notation above, we can define a bilinear map

$$(5.4) \qquad \boldsymbol{K}(\boldsymbol{\xi},\boldsymbol{\eta}) := \sum_{i<j,k<l} \kappa_i^j(\boldsymbol{e}_k,\boldsymbol{e}_l)\xi^{kl}\eta^{ij}, \qquad \boldsymbol{\xi} = \sum_{k<l}\xi^{kl}\boldsymbol{e}_k \wedge \boldsymbol{e}_l, \quad \boldsymbol{\eta} = \sum_{i<j}\eta^{ij}\boldsymbol{e}_i \wedge \boldsymbol{e}_j$$

on $\wedge^2 T_pM$, where $\boldsymbol{e}_j$, κ_i^j...are considered tangent vectors, 2-forms at the fixed point p. In fact, one can show that the definition (5.4) is independent of choice of orthonormal frames. As a immediate conclusion of Corollary 5.3, we have

Lemma 5.4. *$\boldsymbol{K}$ is symmetric.*

Hence, taking Lemma 5.1 into an account, we define the sectional curvature as follows:

Definition 5.5. Let $\Pi_p \subset T_pM$ be a 2-dimensional linear subspace in T_pM. The *sectional curvature* of (M,g) with respect to the plane Π_p is a number

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v}\wedge\boldsymbol{w},\boldsymbol{v}\wedge\boldsymbol{w}),$$

where $\{\boldsymbol{v},\boldsymbol{w}\}$ is an orthonormal basis of Π_p

Remark 5.6. For (not necessarily orthonormal) basis $\{\boldsymbol{x},\boldsymbol{y}\}$ of Π_p, the sectional curvature is expressed as

$$K(\Pi_p) = \frac{\boldsymbol{K}(\boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{x}\wedge\boldsymbol{y})}{\langle\boldsymbol{x}\wedge\boldsymbol{y},\boldsymbol{x}\wedge\boldsymbol{y}\rangle},$$

where $\langle\ ,\ \rangle$ of the right-hand side is the inner product of $\wedge^2 T_pM$ induced from the Riemannian metric.

Remark 5.7. The sectional curvature is a scalar corresponding to a 2-plane in the tangent space T_pM. Hence it can be considered as a function of 2-Grassmannian bundle induced from the tangent bundle TM.

5.3 Curvature Tensor

Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection. Define a trilinear map

$$(5.5) \quad R\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \ni (X,Y,Z) \mapsto R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \in \mathfrak{X}(M).$$

By the properties Lemma 3.6 of the connection and the property (3.4) of the Lie bracket, the following Lemma is obvious.

Lemma 5.8. *For any function $f \in \mathcal{F}(M)$ and vector fields X, Y, $Z \in \mathfrak{X}(M)$,*

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)(fZ) = fR(X,Y)Z$$

holds.

Corollary 5.9. *Assume the vector fields X, Y, Z and $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z} \in \mathfrak{X}(M)$ satisfy $X_p = \widetilde{X}_p$, $Y_p = \widetilde{Y}_p$ and $Z_p = \widetilde{Z}_p$ for a point $p \in M$. Then*

$$\bigl(R(X,Y)Z\bigr)_p = \bigl(R(\widetilde{X},\widetilde{Y})\widetilde{Z}\bigr)_p.$$

In other words, R in (5.5) induces a trilinear map

$$R_p\colon T_pM\times T_pM\times T_pM \to T_pM.$$

Definition 5.10. A trilinear map $R(X,Y)Z$ is called the *curvature tensor* of (M,g). In addition, a quadrilinear map

$$R(X,Y,Z,T) = \langle R(X,Y)Z,T\rangle : \mathfrak{X}(M)^4 \to \mathcal{F}(M)$$

is also called the *curvature tensor*. In fact, $R \in \Gamma(T^*M \otimes T^*M \otimes T^*M \otimes T^*M)$, that is R is $(0,4)$-tensor field, because R induces a quadrilinear map

$$R\colon (T_pM)^4 \to \mathbb{R}$$

for each $p \in M$.

Lemma 5.11. *Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be an orthonormal frame on a domain $U \subset M$, and $K = (\kappa_i^j)$ the curvature form with respect to the frame. Then it holds that*

$$\kappa_i^j(X,Y) = R(X,Y,\boldsymbol{e}_i,\boldsymbol{e}_j)$$

for each (i,j).

So by (5.3), Proposition 5.2, Corollary 5.3 yield

Proposition 5.12.
- $R(X,Y,Z,T) = -R(Y,X,Z,T) = -R(X,Y,T,Z)$,
- $R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0$,
- $R(X,Y,Z,T) = R(Z,T,X,Y)$.

Moreover, the sectional curvature $K(\Pi_p)$ in Definition 5.5 is computed by

$$K(\Pi_p) = \frac{R(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y},\boldsymbol{x})}{\langle \boldsymbol{x},\boldsymbol{x}\rangle \langle \boldsymbol{y},\boldsymbol{y}\rangle - \langle \boldsymbol{x},\boldsymbol{y}\rangle^2}. \tag{5.6}$$

Exercises

5-1 Consider a Riemannian metric

$$g = dr^2 + \{\varphi(r)\}^2\,d\theta^2 \qquad \text{on} \qquad U := \{(r,\theta)\,;\, 0<r<r_0, -\pi<\theta<\pi\},$$

where $r_0 \in (0,+\infty]$ and φ is a positive smooth function defined on $(0,r_0)$ with

$$\lim_{r\to+0}\varphi(r) = 0, \qquad \lim_{r\to+0}\frac{\varphi(r)}{r} = 1.$$

Classify the function φ so that g is of constant sectional curvature.

5-2 Let $M \subset \mathbb{R}^{n+1}$ be an embedded submanifold endowed with the Riemannian metric induced from the canonical Euclidean metric of $\mathbb{R}^{n+1}$. Then the position vector $\boldsymbol{x}(p)$ of $p \in M$ induces a smooth map

$$\boldsymbol{x}\colon M \ni p \longmapsto \boldsymbol{x}(p) \in \mathbb{R}^{n+1},$$

which is an $(n+1)$-tuple of C^∞-functions. Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be an orthonormal frame defined on a domain $U \subset M$. Since $T_pM \subset \mathbb{R}^{n+1}$, we can consider that $\boldsymbol{e}_j$ is a smooth map from $U \to \mathbb{R}^{n+1}$. Take a dual basis (ω^j) to $[\boldsymbol{e}_j]$. Prove that

$$d\boldsymbol{x} = \sum_{j=1}^n \boldsymbol{e}_j\omega^j$$

holds on U. Here, we regard that $d\boldsymbol{x}$ is an $(n+1)$-tuple of differential forms and $\boldsymbol{e}_j$ is an $\mathbb{R}^{n+1}$-valued function for each j.

6 Space forms

6.1 Constant sectional curvature

Let (M, g) be a Riemannian n-manifold, and let

$$\begin{aligned}&\mathrm{Gr}_2(TM) := \cup_p \mathrm{Gr}_2(T_pM),\\ &\qquad\qquad \mathrm{Gr}_2(T_pM) := \text{2-Grassmannian of } T_pM = \{\Pi_p \subset T_pM\,;\ \text{2-dimensional subspace}\}.\end{aligned}$$

The sectional curvature defined in Definition 5.5 is a map $K\colon \mathrm{Gr}_2(TM) \to \mathbb{R}$ such that

$$K(\Pi_p) := \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w}),$$

where $\{\boldsymbol{v}, \boldsymbol{w}\}$ is the orthonormal basis of Π_p.

Fix a point p, and take an orthornormal frame $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_n]$ defined on a neighborhood U of p. Denote by (ω^j), $\Omega = (\omega_i^j)$ and $K = (\kappa_i^j)$ the dual frame, the connection form and the curvature form with respect to the frame $[\boldsymbol{e}_j]$, respectively.

Theorem 6.1. *Assume there exists a real number k such that $K(\Pi_p) = k$ for all 2-dimensional subspace $\Pi_p \in T_pM$ for a fixed p. Then the curvature form is expressed as*

$$\kappa_j^i = k\omega^i \wedge \omega^j.$$

Conversely, the curvature form is written as above, the sectional curvature at p is constant k.

Proof. By the assumption, $k = K(\mathrm{Span}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_j, \boldsymbol{e}_i \wedge \boldsymbol{e}_j) = \kappa_j^i(\boldsymbol{e}_i, \boldsymbol{e}_j)$. Let

$$\boldsymbol{v} := \cos\theta \boldsymbol{e}_i + \sin\theta \boldsymbol{e}_j, \qquad \boldsymbol{w} := \cos\varphi \boldsymbol{e}_l + \sin\varphi \boldsymbol{e}_m$$

where $\{i, j\} \neq \{l, m\}$, and set $\Pi_{\theta,\varphi} := \mathrm{Span}\{\boldsymbol{v}, \boldsymbol{w}\} \subset T_pM$. Then by bilinieartity of the $\wedge$-product on T_pM, it holds that

$$\boldsymbol{v} \wedge \boldsymbol{w} = \cos\theta\cos\varphi \boldsymbol{e}_i \wedge \boldsymbol{e}_l + \cos\theta\sin\varphi \boldsymbol{e}_i \wedge \boldsymbol{e}_m + \sin\theta\cos\varphi \boldsymbol{e}_j \wedge \boldsymbol{e}_l + \sin\theta\sin\varphi \boldsymbol{e}_j \wedge \boldsymbol{e}_m.$$

Since $\{\boldsymbol{v}, \boldsymbol{w}\}$ is an orthonormal basis of $\Pi_{\theta,\varphi}$, bilinieartity and symmetricity of $\boldsymbol{K}$ implies

$$\begin{aligned}k =& K(\Pi_{\theta,\varphi}) = \boldsymbol{K}(\boldsymbol{v} \wedge \boldsymbol{w}, \boldsymbol{v} \wedge \boldsymbol{w})\\ =& \cos^2\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_l) + \cos^2\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_i \wedge \boldsymbol{e}_m)\\ &+ \sin^2\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) + \sin^2\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m)\\ &+ 2\cos^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) + 2\cos\theta\sin\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l)\\ &+ 2\cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l))\\ &+ 2\cos\theta\sin\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + 2\sin^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m)\\ =& k + 2\big(\cos^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) + \cos\theta\sin\theta\cos^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l)\\ &+ \cos\theta\sin\theta\cos\varphi\sin\varphi(\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l))\\ &+ \cos\theta\sin\theta\sin^2\varphi \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \sin^2\theta\cos\varphi\sin\varphi \boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m)\big).\end{aligned} \tag{6.1}$$

So, by letting $\theta = 0$, we have

$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_i \wedge \boldsymbol{e}_m) = 0. \tag{6.2}$$

Similarly, letting $\theta = \pi/2$, $\varphi = 0$ and $\varphi = \pi/2$, we have $\boldsymbol{K}(\boldsymbol{e}_j \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) = 0$. Hence the equality (6.1) implies

$$\boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_l, \boldsymbol{e}_j \wedge \boldsymbol{e}_m) + \boldsymbol{K}(\boldsymbol{e}_i \wedge \boldsymbol{e}_m, \boldsymbol{e}_j \wedge \boldsymbol{e}_l) = 0.$$

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By definition (5.4), this is equivalent to

$$\kappa_j^m(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_j^l(\boldsymbol{e}_i, \boldsymbol{e}_m) = -(\kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_l^j(\boldsymbol{e}_i, \boldsymbol{e}_m)).$$

Then by Proposition 5.2, we have

$$0 = \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) + \kappa_l^j(\boldsymbol{e}_i, \boldsymbol{e}_m) = \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) - \kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) - \kappa_m^j(\boldsymbol{e}_l, \boldsymbol{e}_i) = 2\kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) - \kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l).$$

Exchanging the roles of i and m, it holds that $2\kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) - \kappa_m^j(\boldsymbol{e}_i, \boldsymbol{e}_l) = 0$. So we have

$$\kappa_i^j(\boldsymbol{e}_m, \boldsymbol{e}_l) = 0 \qquad (\text{if } \{i, j\} \neq \{m, l\}). \tag{6.3}$$

On the other hand, (6.2) means that $\kappa_i^j(\boldsymbol{e}_i, \boldsymbol{e}_l) = \kappa_i^j(\boldsymbol{e}_j, \boldsymbol{e}_l) = 0$ when $l \neq i, j$. Summing up, we have

$$\kappa_i^j(\boldsymbol{e}_k, \boldsymbol{e}_l) = \begin{cases} k & (i, j) = (k, l) \\ 0 & \text{otherwise,} \end{cases}$$

proving the theorem. □

We now consider the case that the assumption of Theorem 6.1 holds for each $p \in M$.

Theorem 6.2. *Assume that for each p, there exists a real number $k(p)$ such that $K(\Pi_p) = k(p)$ for any $\Pi_p \in \mathrm{Gr}_2(T_pM)$. Then the function $k\colon M \ni p \to k(p) \in \mathbb{R}$ is constant provided that M is connected.*

Proof. By taking the exterior derivative of $\kappa_i^j = d\omega_i^j + \sum_s \omega_s^j \wedge \omega_i^s$, it holds that

$$\begin{aligned} d\kappa_i^j &= d(d\omega_i^j) + \sum_s \omega_s^j \wedge d\omega_i^s - \sum_s d\omega_s^j \wedge \omega_i^s \\ &= \sum_s \left(\kappa_s^j - \sum_t \omega_t^j \wedge \omega_s^t\right) \wedge \omega_i^s - \sum_s \omega_s^j \wedge \left(\kappa_i^s - \sum_t \omega_t^s \wedge \omega_i^t\right), \end{aligned}$$

and hence we have the identity

$$d\kappa_i^j = \sum_s \left(\kappa_s^j \wedge \omega_i^s - \omega_s^j \wedge \kappa_i^s\right), \tag{6.4}$$

which is known as the *second Bianchi identity*. By our assumption, Theorem 6.1 implies that $\kappa_i^j = k\omega^i \wedge \omega^j$. Then by Lemma 3.17,

$$\begin{aligned} d\kappa_i^j &= d(k\omega^i) \wedge \omega^j - k\omega^i \wedge d\omega^j = dk \wedge \omega^i \wedge \omega^j + kd\omega^i \wedge \omega^j - k\omega^i \wedge d\omega^j \\ &= dk \wedge \omega^i \wedge \omega^j + \sum_s k\omega^s \wedge \omega_s^i \wedge \omega^j - \sum_s k\omega^i \wedge \omega^s \wedge \omega_s^j = dk \wedge \omega^i \wedge \omega^j + d\kappa_i^j \end{aligned}$$

holds for each i and j. Thus, $dk \wedge \omega^i \wedge \omega^j = 0$ for all i and j, which implies $dk = 0$. This equality is independent of choice of orthonormal frames. Since M is connected, k is constant. □

6.2 Space forms

Let (M, g) be a Riemannian n-manifold. A path $\gamma\colon [0, +\infty) \to M$ is said to be a *divergence path* if for any compact subset $K \in M$, there exists $t_0 \in (0, +\infty)$ such that $\gamma([t_0, +\infty)) \subset M \setminus K$. If any divergent path has infinite length, (M, g) is said to be complete.[9] In particular, a compact Riemannian manifold without boundary is automatically complete.

[9] Usually, completeness is defined in terms of geodesics: A Riemannian manifold (M, g) is complete if any geodesics are defined on entire $\mathbb{R}$. The definition here is one of the equivalent conditions of completeness, expressed in the *Hopf-Rinow theorem. cf. MTH.B505.*

Definition 6.3. An n-dimensional *space form* is a complete Riemannian n-manifold of constant sectional curvature.

Example 6.4. The Euclidean n-space is a space form of constant sectional curvature 0. In fact, let $(x^1,\dots,x^n)$ be the canonical Cartesian coordinate system and set $\boldsymbol{e}_j = \partial/\partial x^j$. Then $[\boldsymbol{e}_j]$ is an orthornormal frame defined on the entire $\mathbb{R}^n$, and Propositions 4.1 and 4.2 implies that the connection form $\omega^i_j = 0$. Hence the curvature forms vanish, and then the sectional curvature is identically zero.

So it is sufficient to show completeness. Let $\gamma\colon [0,+\infty) \to \mathbb{R}^n$ be a divergent path. Then for each $r > 0$, there exists $t_0 > 0$ such that $|\gamma(t)| > r$ holds on $[t_0,+\infty)$, equivalently, $|\gamma(t)| \to +\infty$ as $t \to +\infty$. So the length L of the curve γ is

$$L = \lim_{t\to+\infty} \int_0^t |\dot\gamma(\tau)|\,d\tau \geqq \lim_{t\to+\infty} \left|\int_0^t \dot\gamma(\tau)\,d\tau\right| = \lim_{t\to+\infty} |\gamma(t)-\gamma(0)| \geqq \lim_{t\to+\infty} |\gamma(t)| - |\gamma(0)| = +\infty.$$

Here, we used the triangle inequality of integrals for vector-valued functions[10].

6.3 The Hyperbolic spaces

Let $H^n(-c^2)$ be the hyperbolic n-space defined, where c is a non-zero constant:

$$H^n(-c^2) := \left\{ \boldsymbol{x} = (x^0,\dots,x^n) \in \mathbb{R}^{n+1}_1 \;\middle|\; \langle \boldsymbol{x},\boldsymbol{x}\rangle_L = -\frac{1}{c^2}, cx_0 > 0 \right\},$$

where $(\mathbb{R}^{n+1}_1, \langle\ ,\ \rangle_L)$ be the Lorentz-Minkowski $(n+1)$-space. The tangent space $T_{\boldsymbol{x}}H^n(-c^2)$ is the orthogonal complement $\boldsymbol{x}^\perp$ of $\boldsymbol{x}$, and the restriction g_H of the inner product $\langle\ ,\ \rangle_L$ to $T_{\boldsymbol{x}}H^n(-c^2)$ is positive definite. Thus, $(H^n(-c^2), g_H)$ is a Riemannian manifold, called the *hyperbolic n-space*.

Theorem 6.5. *The hyperbolic space $(H^n(-c^2), g_H)$ is of constant sectional curvature $-c^2$.*

Proof. Notice that $H^n(-c^2)$ can be expressed as a graph $x^0 = \frac{1}{c}\sqrt{1 + c^2\left((x^1)^2 + \dots + (x^n)^2\right)}$ defined on the $(x^1,\dots,x^n)$-hyperplane, that is, it is covered by single chart. Then there exists a orthonormal frame field $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ defined on entire $H^n(-c^2)$. Denote by (ω^i), $\Omega = (\omega^j_i)$ and $K = (\kappa^j_i)$ the dual frame, the connection form and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively.

Regarding $T_{\boldsymbol{x}}H^n(-c^2)$ as a linear subspace in $\mathbb{R}^{n+1}_1$, we can consider $\boldsymbol{e}_j$ as a vector-valued function. In addition the position vector $\boldsymbol{x} \in H^n(-c^2)$ can be also regarded as a vector-valued function. Since $T_{\boldsymbol{x}}H^n(-c^2) = \boldsymbol{x}^\perp$,

$$\mathcal{F} := (\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n)\colon H^n(-c^2) \to \mathrm{M}_{n+1}(\mathbb{R}) \qquad \boldsymbol{e}_0 = c\boldsymbol{x} \tag{6.5}$$

gives a pseudo orthornormal frame along $H^n(-c^2)$, that is, $\mathcal{F}^T Y \mathcal{F} = Y$ $(Y := \mathrm{diag}(-1,1,\dots,1))$ holds.

As seen in Exercise 5-2, it holds that

$$d\boldsymbol{e}_0 = c\,d\boldsymbol{x} = c\sum_{j=1}^n \omega^j \boldsymbol{e}_j. \tag{6.6}$$

On the other hand, for each $j = 1,\dots,n$, decompose the vector-valued one form $d\boldsymbol{e}_j$ as

$$d\boldsymbol{e}_j = h_j \boldsymbol{e}_0 + \sum_s \alpha^s_j \boldsymbol{e}_s,$$

[10]See, for example, Theorem A.1.4 in [UY17] for $n = 2$. The idea of the proof works for general n.

where h_j and α_j^s are one forms on $H^n(-c^2)$. Here,

$$h_j = -\langle d\boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L = -d\langle \boldsymbol{e}_j, \boldsymbol{e}_0\rangle_L + \langle \boldsymbol{e}_j, d\boldsymbol{e}_0\rangle_L = c\omega^j,$$

and

$$\alpha_j^s = \langle d\boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L = d\langle \boldsymbol{e}_j, \boldsymbol{e}_s\rangle_L - \langle \boldsymbol{e}_j, d\boldsymbol{e}_s\rangle_L = -\alpha_s^j.$$

Differentiating (6.6), it holds that

$$0 = \frac{1}{c}\, dd\boldsymbol{e}_0 = \sum_j (d\omega^j \boldsymbol{e}_j - \omega^j \wedge d\boldsymbol{e}_j) = \sum_{j,s} \omega^s \wedge \omega_s^j \boldsymbol{e}_j - \sum_{j,s} \omega^j \wedge \alpha_j^s \boldsymbol{e}_s = \sum_j \sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\boldsymbol{e}_j$$

because $\omega^j \wedge \omega^j = 0$. Thus, we have $\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j) = 0$, and then

$$\begin{aligned}
0 &= \left(\sum_s \omega^s \wedge (\omega_s^j - \alpha_s^j)\right)(\boldsymbol{e}_l, \boldsymbol{e}_m) = (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)) - (\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)),\\
0 &= (\omega_j^m(\boldsymbol{e}_l) - \alpha_j^m(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) = -(\omega_m^j(\boldsymbol{e}_l) - \alpha_m^j(\boldsymbol{e}_l)) - (\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)),\\
0 &= (\omega_m^l(\boldsymbol{e}_j) - \alpha_m^l(\boldsymbol{e}_j)) - (\omega_j^l(\boldsymbol{e}_m) - \alpha_j^l(\boldsymbol{e}_m)) = -(\omega_l^m(\boldsymbol{e}_j) - \alpha_l^m(\boldsymbol{e}_j)) + (\omega_l^j(\boldsymbol{e}_m) - \alpha_l^j(\boldsymbol{e}_m)),
\end{aligned}$$

which conclude that $\omega_l^j = \alpha_l^j$. Summing up, we have

$$d\boldsymbol{e}_j = c\omega^j \boldsymbol{e}_0 + \sum_s \omega_j^s \boldsymbol{e}_s. \tag{6.7}$$

Then the frame $\mathcal{F}$ in (6.5) satisfies

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \text{where} \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & c\boldsymbol{\omega}^T \\ c\boldsymbol{\omega} & \Omega \end{pmatrix} \quad \text{and} \quad \boldsymbol{\omega} := (\omega^1, \dots, \omega^n)^T. \tag{6.8}$$

The integrability condition of (6.8) is

$$O = d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = \begin{pmatrix} c^2\boldsymbol{\omega}^T \wedge \boldsymbol{\omega} & c\left(d\boldsymbol{\omega}^T + \omega^T \wedge \Omega\right) \\ c\left(d\boldsymbol{\omega} + \Omega \wedge \boldsymbol{\omega}\right) & d\Omega + \Omega \wedge \Omega + c^2\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T \end{pmatrix}.$$

The lower-right components of the identity above yields

$$\kappa_i^j + c^2\omega^i \wedge \omega^j = 0.$$

Hence the sectional curvature of $(H^n(-c^2), g_H) = -c^2$. □

Remark 6.6. One can show the completeness of $(H^n(-c^2), g_H)$ (cf. MTH.B505). Hence the hyperbolic space is a simply connected space form of constant negative sectional curvature.

6.4 Isometries

A C^∞-map $f\colon M \to N$ between manifolds M and N induces a linear map

$$(df)_p \colon T_pM \ni X \longmapsto (df)_p(X) = \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma(t) \in T_{f(p)}N,$$

where $\gamma\colon (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$ and $\dot\gamma(0) = X$, called the *differential* of f. Since $p \in M$ is arbitrary, this induces a bundle homomorphism $df\colon TM \to TN$.

Definition 6.7. A *vector field on N along a smooth map* $f\colon M \to N$ is a map $X\colon M \to TN$ satisfying $\pi \circ X = f$, where $\pi\colon TN \to N$ is the canonical projection.

Then for each vector field $X \in \mathfrak{X}(M)$, $df(X)$ is a vector field on N along f.

Definition 6.8. A C^∞-map $f\colon M \to N$ between Riemannian manifolds (M,g) and (N,h) is called a *local isometry* if $\dim M = \dim N$ and $f^*h = g$ hold, that is,

$$f^*h(X,Y) := h(df(X), df(Y)) = g(X,Y)$$

holds for X, $Y \in T_pM$ and $p \in M$.

Lemma 6.9. *A local isometry is an immersion.*

Proof. Let $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ be a (local) orthonormal frame of M, where $n = \dim M$. Set $\boldsymbol{v}_j := df(\boldsymbol{e}_j)$ $(j = 1,\dots,n)$ for a smooth map $f\colon (M,g) \to (N,h)$. If f is a local isometry, $[\boldsymbol{v}_1(p),\dots,\boldsymbol{v}_n(p)]$ is an orthonormal system in $T_{f(p)}N$, because

$$h(\boldsymbol{v}_i,\boldsymbol{v}_j) = h(df(\boldsymbol{e}_i), df(\boldsymbol{e}_j)) = f^*h(\boldsymbol{e}_i,\boldsymbol{e}_j) = g(\boldsymbol{e}_i,\boldsymbol{e}_j).$$

Hence the differential $(df)_p$ is of rank n. □

The proof of Lemma 6.9 suggests the following fact:

Corollary 6.10. *A smooth map $f\colon (M,g) \to (N,h)$ is a local isometry if and only if for each $p \in M$,*

$$[\boldsymbol{v}_1,\dots,\boldsymbol{v}_n] := [df(\boldsymbol{e}_1),\dots,df(\boldsymbol{e}_n)]$$

is an orthonormal frame for some orthonormal frame $[\boldsymbol{e}_j]$ on a neighborhood of p.

6.5 Local uniqueness of space forms

Theorem 6.11. *Let $U \subset \mathbb{R}^n$ be a simply connected domain and g a Riemannian metric on U. If the sectional curvature of (U,g) is constant k, there exists a local isometry $f\colon U \to N^n(k)$, where*

$$N^n(k) = \begin{cases} S^n(k) & (k>0) \\ \mathbb{R}^n & (k=0) \\ H^n(k) & (k<0). \end{cases}$$

Proof. Take an orthonormal frame $[\boldsymbol{e}_1,\dots,\boldsymbol{e}_n]$ on U, and let (ω^j), $\varOmega = (\omega_i^j)$ and $K = (\kappa_i^j)$ be the dual frame, the connection form, and the curvature form with respect to $[\boldsymbol{e}_j]$, respectively. Since the sectional curvature is constant k, $\kappa_i^j = k\omega^i \wedge \omega^j$ holds for each (i,j), because of Theorem 6.1.

First, consider the case $k = 0$: In this case, $K = d\varOmega + \varOmega \wedge \varOmega = O$, and then by Theorem 4.5, there exists the unique matrix valued function $\mathcal{F}\colon U \to \mathrm{SO}(n)$ satisfying

$$d\mathcal{F} = \mathcal{F}\varOmega, \qquad \mathcal{F}(p_0) = \mathrm{id},$$

where $p_0 \in U$ is a fixed point. Decompose the matrix $\mathcal{F}$ into column vectors as $\mathcal{F} = [\boldsymbol{v}_1,\dots,\boldsymbol{v}_n]$, and define an $\mathbb{R}^n$-valued one form

$$\boldsymbol{\alpha} := \sum_{j=1}^n \omega^j \boldsymbol{v}_j.$$

Then

$$d\boldsymbol{\alpha} = \sum_{j=1}^n \Bigl(d\omega^j \boldsymbol{v}_j - \omega^j \wedge d\boldsymbol{v}_j\Bigr) = \sum_{j,s} \Bigl(\omega^s \wedge \omega_s^j\Bigr)\boldsymbol{v}_j - \sum_{j,s}\Bigl(\omega^j \wedge \omega_j^s\Bigr)\boldsymbol{v}_s = \boldsymbol{0}.$$

Hence by the Poincaré lemma (Theorem 4.8), there exists a smooth map $f\colon U \to \mathbb{R}^n$ satisfying $df = \boldsymbol{\alpha}$. For such an f, it holds that

$$df(\boldsymbol{e}_s) = \alpha(\boldsymbol{e}_s) = \sum_{j=1}^{n} \omega^j(\boldsymbol{e}_s)\boldsymbol{v}_j = \boldsymbol{v}_s$$

for $s = 1, \dots, n$. Hence $[df(\boldsymbol{e}_1), \dots, df(\boldsymbol{e}_n)] = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ is an orthonormal frame, and then f is a local isometry because Corollary 6.10.

Next, consider the case $k = -c^2 < 0$. We set

$$\widetilde{\Omega} := \begin{pmatrix} 0 & c\boldsymbol{\omega}^T \\ c\boldsymbol{\omega} & \Omega \end{pmatrix}, \qquad \text{where} \qquad \boldsymbol{\omega} = \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}$$

as in (6.8) in Section **??**. Since $\kappa_i^j = k\omega^i \wedge \omega^j = -c^2\omega^i \wedge \omega^j$, $d\widetilde{\Omega} + \widetilde{\Omega} \wedge \widetilde{\Omega} = O$ holds as seen in Section **??**. Hence there exists an matrix valued function $\mathcal{F}\colon U \to \mathrm{M}_{n+1}(\mathbb{R})$ satisfying

$$d\mathcal{F} = \mathcal{F}\widetilde{\Omega}, \qquad \mathcal{F}(p_0) = \mathrm{id}, \tag{6.9}$$

where $p_0 \in U$ is a fixed point. Notice that

$$\widetilde{\Omega}^T Y + Y\widetilde{\Omega} = O \qquad Y = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

holds,

$$d(\mathcal{F}Y\mathcal{F}^T) = \mathcal{F}\widetilde{\Omega}Y\mathcal{F}^T + \mathcal{F}Y\widetilde{\Omega}^T\mathcal{F}^T = \mathcal{F}(\widetilde{\Omega}Y + Y\widetilde{\Omega}^T)\mathcal{F}^T = O.$$

Hence, by the initial condition,

$$\mathcal{F}Y\mathcal{F}^T = Y, \qquad \text{that is,} \qquad (\mathcal{F}Y)^{-1} = \mathcal{F}^T Y.$$

Thus, we have

$$\mathcal{F}^T Y \mathcal{F} = (\mathcal{F}Y)^{-1}\mathcal{F} = Y\mathcal{F}^{-1}\mathcal{F} = Y. \tag{6.10}$$

Decompose $\mathcal{F} = [\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$. Then (6.10) is equivalent to

$$-\langle \boldsymbol{v}_0, \boldsymbol{v}_0 \rangle_L = \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle_L = \dots = \langle \boldsymbol{v}_n, \boldsymbol{v}_n \rangle_L = 1, \qquad \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0 \quad (\text{if } i \neq j). \tag{6.11}$$

In particular, the 0-th component of $\boldsymbol{v}_0$ never vanishes, since

$$-1 = \langle \boldsymbol{v}_0, \boldsymbol{v}_0 \rangle_L = -(v_0^0)^2 + (v_0^1)^2 + \dots + (v_0^n)^2 \qquad \boldsymbol{v}_0 = (v_0^0, v_0^1, \dots, v_0^n)^T.$$

Moreover, by the initial condition $\boldsymbol{v}_0(p_0) = (1, 0, \dots, 0)^T$,

$$v_0^0 > 0 \tag{6.12}$$

holds.

Set $f := \frac{1}{c}\boldsymbol{v}_0$. Then $f\colon U \to \mathbb{R}_1^{n+1}$ is the desired map. In fact, by (6.11) and (6.12),

$$f \in H^n(-c^2) = \left\{ \boldsymbol{x} = (x^0, \dots, x^n)^T \in \mathbb{R}_1^{n+1} \,\middle|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -\frac{1}{c^2}, cx^0 > 0 \right\},$$

and

$$df(\boldsymbol{e}_j) = \frac{1}{c} d\boldsymbol{v}_0(\boldsymbol{e}_j) = \sum_{s=1}^{n} \omega^s(\boldsymbol{e}_j)\boldsymbol{v}_s = \boldsymbol{v}_j.$$

Hence $[\boldsymbol{v}_j] = [\boldsymbol{e}_j]$ is an orthonormal frame because (6.11).

The case $k > 0$ is left as an exercise. □

Exercises

6-1 Prove that the sphere

$$S^3(1) = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,;\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}$$

of radius 1 in the Euclidean 4-space is of constant sectional curvature 1.

6-2 Prove Theorem 6.11 for $k = 1$ and $n = 2$, assuming Exercise 6-1.

Bibliography

[dC92] M. P. do Carmo, *Riemannian geometry*, Birkhäuser, 1992.

[Lee13] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, 2013.

[SG04] J. Lafontaine S. Gallot, D. Hulin, *Riemannian geometry*, Springer-Verlag, 2004.

[Tu17] L. W. Tu, *Differential geometry*, Springer Verlag, 2017.

[UY17] Masaaki Umehara and Kotaro Yamada, *Differential geometry of curves and surfaces*, World Scientific, 2017.

Glossary

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arc-length parameter 弧長径数, 7

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compatibility condition 適合条件, 9
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covariant 共変, 14
curvature tensor 曲率テンソル, 26
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