Advanced Topics in Geometry A1 (MTH.B405)

Overview

Kotaro Yamada
kotaro@math.sci.isct.ac.jp
http://www.official.kotaroy.com/class/2025/geom-a1

Institute of Science Tokyo

2025/04/11

Our Goal

Theorem (The Fundamental Theorem for surface theory)

Let 2-dim Euclidean space (or coordinate plane)

- $U \subset \mathbb{R}^2$ *be a simply connected domain,* → §3
- I *be a positive definite symmetric quadratic form on* U 1st
- II *be a symmetric quadratic form on* U. 2nd fundamental forms §4
- I *and* II *satisfy the Gauss and Codazzi equations.* §5

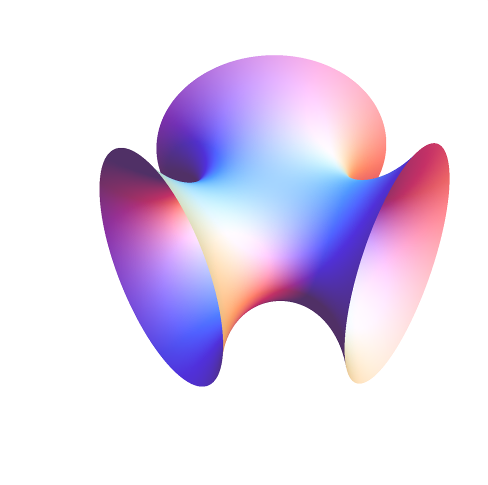
$\Rightarrow$

- *there exists a surface* $\boxed{f : U \to \mathbb{R}^3}$ *with first and second fundamental forms* I *and* II*, resp.* 3-dim Euclidean space
- *such an* f *is unique up to* $\underbrace{\text{orientation preserving isometry of}}_{\text{congruence}}$ $\mathbb{R}^3$.

Surfaces

A surface in $\mathbb{R}^3$:

$\mathbb{R}^3$



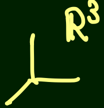
Surfaces

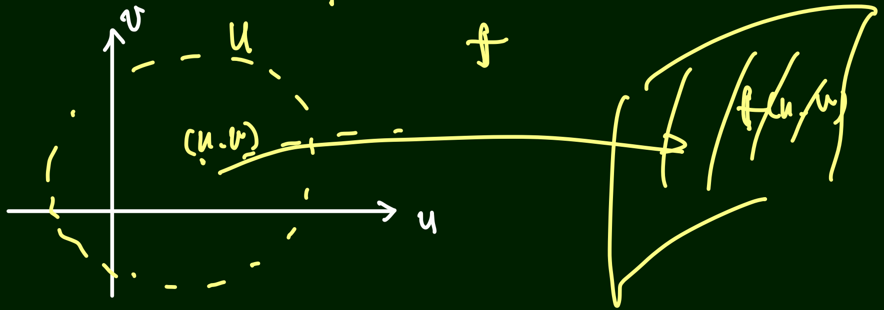
"Minimal surface"

Olympiastadion München (1971); Frei Otto

Expression of Surfaces—Parametrization

- $U \subset \mathbb{R}^2$: a domain ← connected open set.
- $f : U \longrightarrow \mathbb{R}^3$ map





$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $x^2+y^2+z^2=1$

$\|$

$\mathbb{R}^3$

$\cup$

a parametrization of the unit sphere

$$f\colon\ (-\pi,\pi)\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \ni (u,v) \longmapsto f(u,v) = \underline{\underline{\begin{pmatrix} \cos u\cos v \\ \sin u\cos v \\ \sin v \end{pmatrix}}} \in \mathbb{R}^3$$

sphere

map

$\frac{\pi}{2}$

$-\frac{\pi}{2}$

$-\pi$ π

$\xrightarrow{f}$

Parametrization—Example

$$\boldsymbol{g}\colon\ (-\pi,\pi)\times(-\infty,\infty)\ni(\overset{s}{\cancel{u}},\overset{t}{\cancel{v}})\longmapsto \boldsymbol{g}(s,t)=\begin{pmatrix}\cos s\,\mathrm{sech}\,t\\ \sin s\,\mathrm{sech}\,t\\ \tanh t\end{pmatrix}\in\mathbb{R}^3$$

Mercator's world map

$\uparrow t$

$\longrightarrow s$

$\xrightarrow[\boldsymbol{g}]{}$

- $g(s,t) = \begin{pmatrix} \cos s \ \operatorname{sech} t \\ \sin s \ \operatorname{sech} t \\ \tanh t \end{pmatrix}$ $\qquad \operatorname{sech}^2 t + \tanh^2 t = 1$

- Hyperbolic functions (similar to trigonometric functions)

$\cosh t = \dfrac{e^t + e^{-t}}{2} \geq 1$: hyperbolic cosine

$\sinh t = \dfrac{e^t - e^{-t}}{2}$: " sine

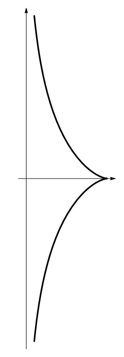
$\tanh t = \dfrac{\sinh t}{\cosh t}$: " tangent

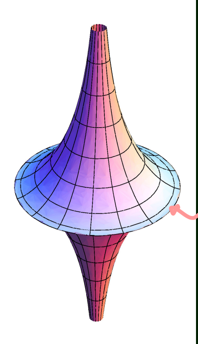
$\operatorname{sech} t = 0 \leq \dfrac{1}{\cosh t} \leq 1$: " secant

Parametrization—Example

smooth functions

$$\boldsymbol{h}\colon (-\pi,\pi)\times(-\infty,\infty)\ni(\overset{s}{\cancel{u}},\overset{t}{\cancel{v}})\longmapsto \boldsymbol{h}(s,t)=\begin{pmatrix}\cos s\,\operatorname{sech} t\\ \sin s\,\operatorname{sech} t\\ t-\tanh t\end{pmatrix}\in\mathbb{R}^3$$





pseudosphere

Beltrami

not smooth

MTH. B406

A Naive Question

Q

What quantities determine the surface?

- a triple of three coordinate functions of $\boldsymbol{f}(u, v)$.

Quantity: indep. on isometries

Isometries: in $\mathbb{R}^3$

translation

$$\boldsymbol{f} \longmapsto A\boldsymbol{f} + \boldsymbol{a} \qquad A \in \mathrm{SO}(3), \boldsymbol{a} \in \mathbb{R}^3$$

rotation

Isometries of $\mathbb{R}^3$

$\langle x, y \rangle$: The inner product

$\|x\| = \sqrt{\langle x, x \rangle}$: the norm

$d(x, y) := \|y - x\|$: the (Euclidean) distance.

Def : A map $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is isometry

$\Leftrightarrow$ $d(x, y) = d(\varphi(x), \varphi(y))$ for all $x, y \in \mathbb{R}^3$

Theorem φ is isometry

translation

$\Leftrightarrow$ $\varphi(x) = Ax + a$

$\boxed{A^T A = id}$

$a \in \mathbb{R}^3$ $A \in O(3) := \{3 \times 3 \text{ orthogonal matrices}\}$

rotation in $\mathbb{R}^3$

$SO(3) := \{A \in O(3) ; \det A = 1\}$

special orth.

A Naive Question

mapped by isometry

Q

What quantities determine the surface up to "congruence"?

integrable conditions $\longrightarrow$ $3 = 6 - 3$

ES

: 6 functions.

$3 \lneq 6$

A

over determined system.

The first and second fundamental forms.

$\cdot f$: 3 functions

The Fundamental Theorem

Theorem (The Fundamental Theorem for surface theory)

- *$U \subset \mathbb{R}^2$ be a simply connected domain,*
- *I be a positive definite symmetric quadratic form on U*
- *II be a symmetric quadratic form on U.*
- *I and II satisfy the Gauss and Codazzi equations.*

$\Rightarrow$

- *$\exists$a surface $f : U \to \mathbb{R}^3$ with I and II,*
- *f is unique up to orientation preserving isometry.*