Advanced Topics in Geometry A1 (MTH.B405)

Overview

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Commutativity of Partial Derivatives

Theorem

(x. y)

- $f : U \to \mathbb{R}$: *a function defined on a domain* $U \subset \mathbb{R}^2$
- $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y}$ *exist and continuous on* U

$\Rightarrow$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)} \qquad p = (u, v) \in U.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}, \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

the mean value theorem (for functions of 1-variable)

(cf.)

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

special case !!

Commutativity of Partial Derivatives

Corollary ("easy to explain" version)

$f : U \to \mathbb{R}$: *a function of class C^2 defined on a domain $U \subset \mathbb{R}^2$*

$\Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

class C^1: $\exists f_x$ $\exists f_y$ & continuous

class C^2: f_x & f_y : of class C^1

$$\underbrace{\exists f_{xx} \quad \exists f_{xy} \quad \exists f_{yx} \quad \exists f_{yy}}_{\text{continuous}}$$

Functions of Class C^r

Definition

A function f defined on a domain $U \subset \mathbb{R}^2$ is said to be

- of class C^0 if it is continuous on U,
- of class C^1 if there exists a partial derivative f_x and f_y on U, and both of them are continuous,
- of class C^r $(r = 2, 3, \dots)$ if it is of class C^{r-1} and all of the $(r-1)$-st partial differentials are of class C^1, and
- of class C^∞ if it is of class C^r for arbitrary non-negative integer r.

Commutativity as a property of C^∞-functions

Theorem

$f : U \to \mathbb{R}$: *of class C^∞* $\Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

We assume here that all functions are of class C^∞ unless otherwise specified.

Our context: partial derivatives always commute.

Example—Harmonicity of holomorphic functions

- $w = f(z)$: a function of complex variable.
- Taking real and imaginary parts of z and w:

$$z = u + \sqrt{-1}v, \qquad w = x + \sqrt{-1}y$$

- Then the function f is decomposed to two real-valued function

$$x = x(u, v), \qquad y = y(u, v)$$

$$\exists \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

h complex number

Theorem (Cauchy-Riemann equation)

f is differentiable in complex variable z $\Leftrightarrow$ $\boxed{\begin{cases} x_u &= y_v \\ x_v &= -y_u \end{cases}}$

C-R

Example—Harmonicity of holomorphic functions

Theorem (Cauchy-Riemann equation)

$x + \sqrt{-1}y = f(u + \sqrt{-1}v)$ *is differentiable* $\Leftrightarrow \begin{cases} x_u &= y_v \\ x_v &= -y_u \end{cases}$.

Corollary

Laplacian

$$\Delta x := x_{uu} + x_{vv} = 0, \qquad \Delta y := y_{uu} + y_{vv} = 0.$$

$$\begin{aligned} \Delta x &= x_{uu} + x_{vv} \\ &= (x_u)_u + (x_v)_v = (y_v)_u + (-y_u)_v \\ &= \underline{y_{vu}} - \underline{y_{uv}} = 0 \end{aligned}$$

x is a harmonic function.

Commutativity in terms of differential forms

f, a, b, c,…: C^∞-functions on a domain $U \subset \mathbb{R}^2$. $(x.y)$

- $\alpha = a\,dx + b\,dy$: differential 1-form 1 form
- $\omega = c\,\underbrace{dx \wedge dy}_{\text{wedge}}$: differential 2-form 2 form

Definition

- For a function f,

1-form

$$df := f_x\,dx + f_y\,dy.$$

total differential

- For a 1-form $\alpha = a\,dx + b\,dy$, exterior derivative

rules:

$$d\alpha := (b_{\cancel{y}\,x} - a_{\cancel{x}\,y})\,dx \wedge dy \quad \left(= da \wedge dx + db \wedge dy \right).$$

$$\left(\begin{array}{l} dx \wedge dy = -dy \wedge dx \\ dx \wedge dx = dy \wedge dy = 0 \end{array} \right)$$

$(a_x dx + a_y dy) \wedge dx$
$+ (b_x dx + b_y dy) \wedge dy$

Commutativity in terms of differential forms

f, a, b, c,…: C^∞-functions on a domain $U \subset \mathbb{R}^2$.

- $\alpha = a\,dx + b\,dy$: differential 1-form
- $\omega = c\,dx \wedge dy$: differential 2-form

Definition

- For a function f,

$$df := f_x\,dx + f_y\,dy.$$

- For a 1-form $\alpha = a\,dx + b\,dy$,

$$d\alpha := (b_{\cancel{y}}^{x} - a_{\cancel{x}}^{y})\,dx \wedge dy \qquad \Big(= da \wedge dx + db \wedge dy\Big).$$

Commutativity in terms of differential forms

Definition

- f: a function $\Rightarrow df = f_x\,dx + f_y\,dy$.
- $\alpha = a\,dx + b\,dy$: a 1-form $d\alpha := \underline{(b_{\cancel{y}}{}^{x} - a_{\cancel{x}}{}^{y})\,dx \wedge dy}$

Lemma

$ddf = 0$ for any function f.

- $d(df) = d(f_x dx + f_y dy)$

  $= (f_{yx} - f_{xy})\, dx \wedge dy$

  $= 0$

Poincaré Lemma

no holes

Theorem (Poincaré lemma)

- $U \subset \mathbb{R}^2$: *a simply connected domain*
- α*: 1-form on* U.
- $d\alpha = 0$

$\Rightarrow$

- $\exists f$ *such that* $df = \alpha$.

$d(df) = 0$

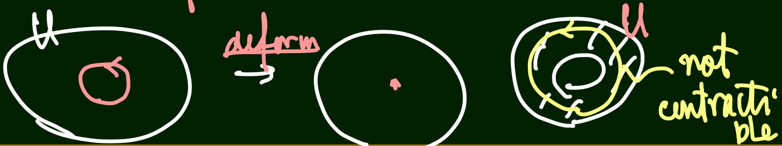
i.e. if $\alpha = df$

$\Rightarrow d\alpha = 0$

$\Leftarrow$

Proof: §3 (def. of simple connectedness)

all loops in U is contractible



Exercise 1-1

Problem (Ex. 1-1)

Let $f(x, y) := e^{ax} \cos y$, *where* a *is constant. Find a function* $g(x, y)$ *satisfying*

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0, 0) = 0.$$

Hint : What value of 'a' admits such 'g'.

Exercise 1-2

Problem (Ex. 1-2)

minus

- $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\} \subset \mathbb{R}^2$,
- $\alpha = a(x,y)\,dx + b(x,y)\,dy := \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy.$
- $\mathrm{P} = (r\cos\theta, r\sin\theta) \in U \ (r > 0,\ 0 < \theta < \pi)$
- $\begin{array}{ll} c_1(t) := \big(x_1(t), y_1(t)\big) = (\cos t, \sin t) & (0 \leqq t \leqq \theta), \\ c_2(s) := \big(x_2(s), y_2(s)\big) = (s\cos\theta, s\sin\theta) & (1 \leqq s \leqq r) \end{array},$

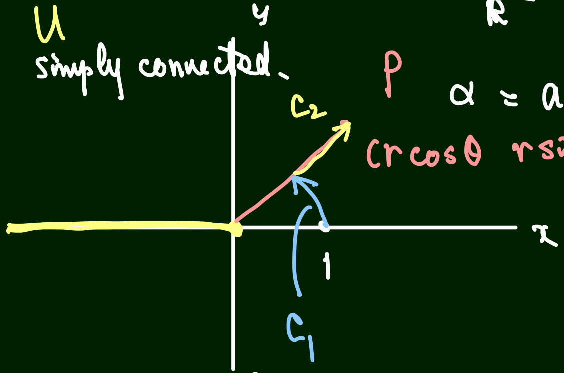
Compute the line integral

$$\int_{c_1 \cup c_2} \alpha := \int_0^\theta \left(\alpha(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + \beta(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt\right) + \int_1^r \left(\alpha(x_2(s), y_2(s))\,\frac{dx_2}{ads}\,ds + \beta(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds\right).$$

U

simply connected.

$\mathbb{R}^2$



P $\alpha = a\,dx + b\,dy$

$(r\cos\theta,\ r\sin\theta)$ $r > 0$

$\theta \in (0, \pi)$

def. of the line integral

$$\int_{C_1 \cup C_2} \alpha = \int_0^{\theta} \left(a \frac{dx_1}{dt} + b \frac{dx_2}{dt} \right) dt$$

$$+ \int_1^{r} \left(a \frac{dx_1}{ds} + b \frac{dx_2}{ds} \right) ds$$

Homeworks

- Solve a problem eithere 1-1 or 1-2 (2 points),
- Present a question on the contents of the lecture, or to point out error(s) in the lecture note/the lecture (up to 3 points).

Deadline: 15. April, 2025, 10:00 JST.

- Submit via LMS
- Format: PDF, 2 pages.
- Use Homework sheet on LMS