Advanced Topics in Geometry A1 (MTH.B405)

Overview

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2025/04/11 (2023/04/25 訂正)

Commutativity of Partial Derivatives

Theorem

- $f\colon U \to \mathbb{R}$: *a function defined on a domain* $U \subset \mathbb{R}^2$
- $f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x}$, $f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y}$ *exist and continuous on* U

$\Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p) \qquad p = (u, v) \in U.$$

Commutativity of Partial Derivatives

Corollary

$f\colon U \to \mathbb{R}$: a function of class C^2 defined on a domain $U \subset \mathbb{R}^2$

$\Rightarrow$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Functions of Class C^r

Definition

A function f defined on a domain $U \subset \mathbb{R}^2$ is said to be

- <u>of class C^0</u> if it is continuous on U,
- <u>of class C^1</u> if there exists a partial derivative f_x and f_y on U, and both of them are continuous,
- <u>of class C^r</u> $(r = 2, 3, \dots)$ if it is of class C^{r-1} and all of the $(r-1)$-st partial differentials are of class C^1, and
- <u>of class C^∞</u> if it is of class C^r for arbitrary non-negative integer r.

Commutativity as a property of C^∞-functions

Theorem

$f\colon U \to \mathbb{R}$: <u>*of class* C^∞</u> $\Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

We assume here that all functions are of class C^∞ unless otherwise specified.

Example—Harmonicity of holomorphic functions

- $w = f(z)$: a function of complex variable.
- Taking real and imaginary parts of z and w:

$$z = u + \sqrt{-1}v, \qquad w = x + \sqrt{-1}y$$

- Then the function f is decomposed to two real-valued function

$$x = x(u, v), \qquad y = y(u, v)$$

Theorem (Cauchy-Riemann equation)

f is differentiable in complex variable z $\Leftrightarrow \begin{cases} x_u &= \ \ y_v \\ x_v &= -y_u \end{cases}$

Example—Harmonicity of holomorphic functions

Theorem (Cauchy-Riemann equation)

$x + \sqrt{-1}y = f(u + \sqrt{-1}v)$ *is differentiable* $\Leftrightarrow \begin{cases} x_u &= y_v \\ x_v &= -y_u \end{cases}$

Corollary

$$\Delta x := x_{uu} + x_{vv} = 0, \qquad \Delta y := y_{uu} + y_{vv} = 0.$$

Commutativity in terms of differential forms

f, a, b, c,...: C^∞-functions on a domain $U \subset \mathbb{R}^2$.

- $\alpha = a\,dx + b\,dy$: differential 1-form
- $\omega = c\,dx \wedge dy$: differential 2-form

Definition

- For a function f,
$$df := f_x\,dx + f_y\,dy.$$
- For a 1-form $\alpha = a\,dx + b\,dy$,
$$d\alpha := (b_y - a_x)\,dx \wedge dy \qquad \Big(= da \wedge dx + db \wedge dy\Big).$$

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Commutativity in terms of differential forms

Definition

- f: a function $\Rightarrow df = f_x\,dx + f_y\,dy$.
- $\alpha = a\,dx + b\,dy$: a 1-form $d\alpha := (b_y - a_x)\,dx \wedge dy$

Lemma

$ddf = 0$ for any function f.

Poincaré Lemma

Theorem (Poincaré lemma)

- $U \subset \mathbb{R}^2$: *a simply connected domain*
- α: 1*-form on* U.
- $d\alpha = 0$

$\Rightarrow$

- $\exists f$ *such that* $df = \alpha$.

Exercise 1-1

Problem (Ex. 1-1)

Let $f(x,y) := e^{ax}\cos y$*, where* a *is constant. Find a function* $g(x,y)$ *satisfying*

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

Exercise 1-2

Problem (Ex. 1-2)

- $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\} \subset \mathbb{R}^2$,
- $\alpha = a(x,y)\,dx + b(x,y)\,dy := \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy.$
- $\mathrm{P} = (r\cos\theta, r\sin\theta) \in U \;\; (r > 0,\; 0 < \theta < \pi)$
- $\begin{array}{ll} c_1(t) := \bigl(x_1(t), y_1(t)\bigr) = (\cos t, \sin t) & (0 \leqq t \leqq \theta), \\ c_2(s) := \bigl(x_2(s), y_2(s)\bigr) = (s\cos\theta, s\sin\theta) & (1 \leqq s \leqq r) \end{array}'$

Compute the line integral

$$\begin{aligned}\int_{c_1 \cup c_2} \alpha := &\int_0^\theta \left(\alpha(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + \beta(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt\right) \\ &+ \int_1^r \left(\alpha(x_2(s), y_2(s))\,\frac{dx_2}{ads}\,ds + \beta(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds\right).\end{aligned}$$

Homeworks

- Solve a problem eithere 1-1 or 1-2 (2 points),
- Present a question on the contents of the lecture, or to point out error(s) in the lecture note/the lecture (up to 3 points).

Deadline: 15. April, 2025, 10:00 JST.

- Submit via LMS
- Format: PDF, 2 pages.
- Use Homework sheet on LMS