

Advanced Topics in Geometry A1 (MTH.B405)

Ordinary Differential Equations

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Exercise 1-1

Problem (Ex. 1-1)

Let $f(x, y) = e^{ax} \cos y$, where a is a constant. Find a function $g(x, y)$ satisfying

$$g_x = -f_y, \quad g_y = f_x, \quad g(0, 0) = 0.$$

Exercise 1-1

$$f(x, y) = e^{ax} \cos y, \quad g_x = -f_y, \quad g_y = f_x, \quad g(0, 0) = 0.$$

Exercise 1-2

Problem (Ex. 1-2)

α : a 1-form on $U = \mathbb{R}^2 \setminus \{(t, 0) ; t \leq 0\}$ as

$$\alpha = a(x, y) dx + b(x, y) dy := \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

$P = (r \cos \theta, r \sin \theta) \in U$ ($r > 1$, $0 < \theta < \pi$),

$$c_1(t) := (x_1(t), y_1(t)) = (\cos t, \sin t) \quad (0 \leq t \leq \theta),$$

$$c_2(s) := (x_2(s), y_2(s)) = (s \cos \theta, s \sin \theta) \quad (1 \leq s \leq r),$$

$$\begin{aligned} \int_{c_1 \cup c_2} \alpha &:= \int_0^\theta \left(\textcolor{blue}{a}(x_1(t), y_1(t)) \frac{dx_1}{dt} dt + \textcolor{blue}{b}(x_1(t), y_1(t)) \frac{dy_1}{dt} dt \right) \\ &\quad + \int_1^r \left(\textcolor{blue}{a}(x_2(s), y_2(s)) \frac{dx_2}{ds} ds + \textcolor{blue}{b}(x_2(s), y_2(s)) \frac{dy_2}{ds} ds \right) = ? \end{aligned}$$

Exercise 1-2; Differential Forms

- ▶ $\alpha = a(x, y) dx + b(x, y) dy$: a 1-form
- ▶ $\beta = c(x, y) dx \wedge dy$: a 2-form

$$df := f_x dx + f_y dy \quad f = f(x, y): \text{a function}$$
$$d\alpha := (b_x - a_y) dx \wedge dy \quad \alpha = a dx + b dy: \text{a 1-form.}$$

Exercise 1-2; Differential Forms

The Exterior Product:

$$(1\text{-form}) \wedge (1\text{-form}) = (2\text{-form}); \quad \text{bilinear, skew-symmetric}$$

The Exterior Derivative:

$$d(f\alpha) = df \wedge \alpha + f d\alpha, \quad d(dx) = d(dy) = 0$$

(f: function, α : 1-form)

Exercise 1-2; Poincaré lemma

$$\alpha = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \quad \Rightarrow \quad d\alpha = 0$$

Theorem (Poincaré lemma)

- ▶ $U \subset \mathbb{R}^2$: a simply connected domain
 - ▶ α : 1-form on U .
 - ▶ $d\alpha = 0$
- \Rightarrow
- ▶ $\exists f$ such that $df = \alpha$.

Exercise 1-2; Line Integral (1)

$$U = \mathbb{R}^2 \setminus \{(t, 0) ; t \leq 0\}, P = (r \cos \theta, r \sin \theta)$$

$$\alpha = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$c_1(t) := (x_1(t), y_1(t)) = (\cos t, \sin t) \quad (0 \leq t \leq \theta),$$

$$\int_{c_1} \alpha := \int_0^\theta \left(\textcolor{red}{a}(x_1(t), y_1(t)) \frac{dx_1}{dt} dt + \textcolor{blue}{b}(x_1(t), y_1(t)) \frac{dy_1}{dt} dt \right)$$

Exercise 1-2; Line Integral (2)

$$U = \mathbb{R}^2 \setminus \{(t, 0) ; t \leq 0\}, P = (r \cos \theta, r \sin \theta)$$

$$\alpha = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

$$c_2(s) := (x_2(s), y_2(s)) = (s \cos \theta, s \sin \theta) \quad (1 \leq s \leq r),$$

$$\int_{c_2} \alpha := \int_1^r \left(\textcolor{blue}{a}(x_2(s), y_2(s)) \frac{dx_2}{ds} ds + \textcolor{blue}{b}(x_2(s), y_2(s)) \frac{dy_2}{ds} ds \right).$$

Exercise 1-2

$$P = (x, y) = (r \cos \theta, r \sin \theta)$$

$$\int_{c_1 \cup c_2} \alpha = \theta =$$

$$\Rightarrow df = \alpha$$

Ordinary Differential Equations

$$\frac{d}{dt} \mathbf{x}(t) = f(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (*)$$

- ▶ Existence
- ▶ Uniqueness
- ▶ Regularity on initial conditions and parameters

Example

$$\frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t), \quad x(0) = x_0.$$

Example

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos \omega t + \frac{y_0}{\omega} \sin \omega t \\ -x_0 \omega \sin \omega t + y_0 \cos \omega t \end{pmatrix}$$

Example

$$\frac{dx}{dt} = t(1 + x^2), \quad x(0) = 0.$$