Advanced Topics in Geometry A1 (MTH.B405)

Ordinary Differential Equations

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Exercise 1-1

Problem (Ex. 1-1)

Let $f(x,y) = e^{ax}\cos y$*, where* a *is a constant. Find a function* $g(x,y)$ *satisfying*

$$g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

Exercise 1-1

$$f(x,y) = e^{ax}\cos y, \qquad g_x = -f_y, \qquad g_y = f_x, \qquad g(0,0) = 0.$$

Exercise 1-2

Problem (Ex. 1-2)

α: *a* 1*-form on* $U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$ *as*

$$\alpha = a(x,y)\,dx + b(x,y)\,dy := \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy.$$

$\mathrm{P} = (r\cos\theta, r\sin\theta) \in U \ (r > 1,\ 0 < \theta < \pi)$,

$$\begin{aligned} c_1(t) &:= \big(x_1(t), y_1(t)\big) = (\cos t, \sin t) && (0 \leqq t \leqq \theta), \\ c_2(s) &:= \big(x_2(s), y_2(s)\big) = (s\cos\theta, s\sin\theta) && (1 \leqq s \leqq r), \end{aligned}$$

$$\begin{aligned} \int_{c_1 \cup c_2} \alpha := &\int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right) \\ &+ \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right) = \text{?} \end{aligned}$$

Exercise 1-2; Differential Forms

- $\alpha = a(x, y)\, dx + b(x, y)\, dy$: a 1-form
- $\beta = c(x, y)\, dx \wedge dy$: a 2-form

$$
\begin{aligned}
df &:= f_x\, dx + f_y\, dy & \qquad & f = f(x, y)\text{: a function} \\
d\alpha &:= (b_x - a_y)\, dx \wedge dy & \qquad & \alpha = a\, dx + b\, dy\text{: a 1-form.}
\end{aligned}
$$

Exercise 1-2; Differential Forms

The Exterior Product:

$$(1\text{-form}) \wedge (1\text{-form}) = (2\text{-form}); \qquad \text{bilinear}, \text{skew-symmetric}$$

The Exterior Derivative:

$$d(f\alpha) = df \wedge \alpha + f\, d\alpha, \qquad d(dx) = d(dy) = 0$$

$$(f\colon \text{ function}, \alpha\colon \text{ 1-form})$$

Exercise 1-2; Poincaré lemma

$$\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \qquad \Rightarrow \qquad d\alpha = 0$$

Theorem (Poincaré lemma)

- $U \subset \mathbb{R}^2$: *a simply connected domain*
- α: 1-*form on* U.
- $d\alpha = 0$

$\Rightarrow$

- $\exists f$ *such that* $df = \alpha$.

Exercise 1-2; Line Integral (1)

$U = \mathbb{R}^2 \setminus \{(t,0)\,;\, t \leqq 0\}$, $\mathrm{P} = (r\cos\theta, r\sin\theta)$

$$\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$c_1(t) := \bigl(x_1(t), y_1(t)\bigr) = (\cos t, \sin t) \qquad (0 \leqq t \leqq \theta),$$

$$\int_{c_1} \alpha := \int_0^\theta \left(a(x_1(t), y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t), y_1(t))\,\frac{dy_1}{dt}\,dt \right)$$

Exercise 1-2; Line Integral (2)

$U = \mathbb{R}^2 \setminus \{(t, 0)\,;\, t \leqq 0\}$, $\mathrm{P} = (r\cos\theta, r\sin\theta)$

$$\alpha = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$c_2(s) := \big(x_2(s), y_2(s)\big) = (s\cos\theta, s\sin\theta) \qquad (1 \leqq s \leqq r),$$

$$\int_{c_2} \alpha := \int_1^r \left(a(x_2(s), y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s), y_2(s))\,\frac{dy_2}{ds}\,ds \right).$$

Exercise 1-2

$\mathrm{P} = (x, y) = (r\cos\theta, r\sin\theta)$

$$\int_{c_1 \cup c_2} \alpha = \theta =$$

$\Rightarrow df = \alpha$

Ordinary Differential Equations

$$\frac{d}{dt}\boldsymbol{x}(t) = f\bigl(t, \boldsymbol{x}(t)\bigr), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{$*$}$$

- Existence
- Uniqueness
- Regularity on initial conditions and parameters

Example

$$\frac{d}{dt}x(t) = f(t, x(t)) = \lambda x(t), \qquad x(0) = x_0.$$

Example

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 \cos \omega t + \frac{y_0}{\omega} \sin \omega t \\ -x_0 \omega \sin \omega t + y_0 \cos \omega t \end{pmatrix}$$

Example

$$\frac{dx}{dt} = t(1 + x^2), \qquad x(0) = 0.$$