# Advanced Topics in Geometry A1 (MTH.B405) Ordinary Differential Equations

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# Linear ordinary differential equations

$$\frac{d}{dt}\boldsymbol{x}(t) = A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t),$$

#### Global Existence

# Linear ordinary differential equations in matrix forms

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0,$$

## Preliminaries

### Proposition (Prop. 2.8)

Assume two  $C^{\infty}$  matrix-valued functions X(t) and  $\Omega(t)$  satisfy

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Then

$$\det X(t) = (\det X_0) \exp \int_{t_0}^t \operatorname{tr} \Omega(\tau) \, d\tau.$$

In particular, if  $X_0 \in GL(n, \mathbb{R})$ , then  $X(t) \in GL(n, \mathbb{R})$  for all t.

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

Corollary (Cor. 2.9)

If  $\operatorname{tr} \Omega(t) = 0$ , then  $\det X(t)$  is constant. In particular, if  $X_0 \in \operatorname{SL}(n, \mathbb{R})$ , X is a function valued in  $\operatorname{SL}(n, \mathbb{R})$ .

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = X_0.$$

### Proposition (Prop. 2.10)

Assume  $\Omega^T + \Omega = O$ . If  $X_0 \in O(n)$  (resp.  $X_0 \in SO(n)$ ), then  $X(t) \in O(n)$  (resp.  $X(t) \in SO(n)$ ) for all t.

### Proposition (Prop. 2.12)

Let  $\Omega(t)$  be a  $C^{\infty}$ -function valued in  $M_n(\mathbb{R})$  defined on an interval I. Then for each  $t_0 \in I$ , there exists the unique matrix-valued  $C^{\infty}$ -function  $X(t) = X_{t_0, id}(t)$  such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \qquad X(t_0) = \mathrm{id}\,.$$

## Linear ordinary differential equations.

## Corollary (Cor. 2.13)

There exists the unique matrix-valued  $C^{\infty}$ -function  $X_{t_0,X_0}(t)$  defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t), \quad X(t_0) = X_0 \quad \left(X(t) := X_{t_0, X_0}(t)\right)$$

In particular,  $X_{t_0,X_0}(t)$  is of class  $C^{\infty}$  in  $X_0$  and t.

### Proposition (Prop. 2.14)

Let  $\Omega(t)$  and B(t) be matrix-valued  $C^{\infty}$ -functions defined on I. Then for each  $t_0 \in I$  and  $X_0 \in M_n(\mathbb{R})$ , there exists the unique matrix-valued  $C^{\infty}$ -function defined on I satisfying

$$\frac{dX(t)}{dt} = X(t)\Omega(t) + B(t), \qquad X(t_0) = X_0.$$

### Theorem (Thm. 2.15)

Let I and U be an interval and a domain in  $\mathbb{R}^m$ , respectively, and let  $\Omega(t, \alpha)$  and  $B(t, \alpha)$  be matrix-valued  $C^{\infty}$ -functions defined on  $I \times U$  ( $\alpha = (\alpha_1, \ldots, \alpha_m)$ ). Then for each  $t_0 \in I$ ,  $\alpha \in U$  and  $X_0 \in M_n(\mathbb{R})$ , there exists the unique matrix-valued  $C^{\infty}$ -function  $X(t) = X_{t_0, X_0, \alpha}(t)$  defined on I such that

$$\frac{dX(t)}{dt} = X(t)\Omega(t, \boldsymbol{\alpha}) + B(t, \boldsymbol{\alpha}), \qquad X(t_0) = X_0.$$
 (1)

Moreover,

 $I \times I \times M_n(\mathbb{R}) \times U \ni (t, t_0, X_0, \boldsymbol{\alpha}) \mapsto X_{t_0, X_0, \boldsymbol{\alpha}}(t) \in M_n(\mathbb{R})$ 

is a  $C^{\infty}$ -map.

## Application to Space Curves

 $\gamma: I \rightarrow \mathbb{R}^3$ : a space curve parametrized by the arclength.
*e* = *γ*<sup>'</sup>

- $\kappa = |e'|$ ; we assume  $\kappa > 0$  (the curvature)
- $\blacktriangleright$   $n=e'/\kappa$  (the principal normal)
- $\blacktriangleright \ b = e \times n \ ({\sf the \ binormal})$
- $ightarrow au = -m{b}' \cdot m{n}$  (the torsion)

# Frenet-Serret

$$\mathcal{F} := (\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}) \colon I \to \mathrm{SO}(3) \colon \text{the Frenet Frame}$$
$$\frac{d\mathcal{F}}{ds} = \mathcal{F}\Omega, \qquad \Omega = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}.$$

## Theorem (Thm. 2.17)

Let  $\kappa(s)$  and  $\tau(s)$  be  $C^{\infty}$ -factions defined on an interval Isatisfying  $\kappa(s) > 0$  on I. Then there exists a space curve  $\gamma(s)$  parametrized by arc-length whose curvature and torsion are  $\kappa$  and  $\tau$ , respectively. Moreover, such a curve is unique up to transformation  $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b} \ (A \in \mathrm{SO}(3), \ \boldsymbol{b} \in \mathbb{R}^3)$  of  $\mathbb{R}^3$ .

## Problem (Ex. 2-1)

#### Find the maximal solution of the initial value problem

$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

Exercise 2-2

## Problem (Ex. 2-2)

Let x = x(t) be the maximal solution of an initial value problem of differential equation

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

## Problem (Ex. 2-3)

Find an explicit expression of a space curve  $\gamma(s)$  parametrized by the arc-length s, whose curvature  $\kappa$  and torsion  $\tau$  satisfy

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$