April 25, 2025 (Corrected: April 25, 2025)
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Info. Sheet 2; Advanced Topics in Geometry A1 (MTH.B405)

Informations

- The classes on May 2nd will run on Tuesday's program. Next lecture of this class will be May 9th.
- The deadline of today's homework is 30th of April, Wednesday, 10:00JST, since Tuesday is a national holiday.

Corrections

- Lecture note, page 1, line 11: $d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})$ ⇒ $d\bigl(f(\boldsymbol{x}), f(\boldsymbol{y})\bigr)$
- Lecture note, page 1, line −8: (1.3) → (1.5)
- Lecture note, page 1, line −4: necessary to mathematical ⇒ necessary for mathematical
- Lecture note, page 2, line −10: $\partial^2 f)\partial x\partial y)$ ⇒ $\partial^2 f(\partial x\partial y)$
- Lecture note, page 3, line 1: Since f_y exists ⇒ $(f_x)_y$ exists
- Lecture note, page 3, line 2: mean curvature ⇒ mean value
- Lecture note, page 4, the first line of Exercise 1-1: a is constant ⇒ a is a constant.
- Lecture note, page 4, the 3rd line of Exercise 1-2: $r > 0$ ⇒ $r > 1$
- Lecture note, page 4, 2 lines of the bottom of Exercise 1-2: α, β in the right-hand side ⇒ a, b, $d\frac{x_1}{dt}$, etc. ⇒ $\frac{dx_1}{dt}$, etc., that is the correct formula is:

$$\int_{c_1\cup c_2}\alpha := \int_0^\theta \left(a(x_1(t),y_1(t))\,\frac{dx_1}{dt}\,dt + b(x_1(t),y_1(t))\,\frac{dy_1}{dt}\,dt\right) + \int_1^r \left(a(x_2(s),y_2(s))\,\frac{dx_2}{ds}\,ds + b(x_2(s),y_2(s))\,\frac{dy_2}{ds}\,ds\right).$$

Students' comments

- 毎回 zoom 録画 がアップされるのでしょうか.
  Will zoom recordings be uploaded each time? **Lecturer's comment** そのつもりです. Yes.
- 偏微分の入れ替えはとてもよく使うのに，学部 1 年のとき以来証明を見てませんでした.
  I haven't seen the proof of the commutativity since my first year of undergrad, even though I use it very often. **Lecturer's comment** それが普通だと思います. I think that's normal
- 可能ならアルファベットをブロック体で書いてほしいです（何を書いているかわからないことがある）
  I would like the alphabet written in block letters if possible. (Sometimes it's hard to tell what you're writing.) **Lecturer's comment** 努力します. I'll try.
- 定理のカウンターが連番になっていてありがたい.
  I appreciate that the theorem counters are sequentially numbered. **Lecturer's comment** 山田のスタイルはこれ. This is my style.
- 授業の資料が豊富で助かります. 1–2Q の間よろしくお願いします.
  Thank you very much for your help with the class materials, and I look forward to working with you during the 1–2Q. **Lecturer's comment** こちらこそ. It's I who should say so
- 第一基本形式と第二基本形式について学部の講義で習ったものの忘れてしまったので復習したいと思いました.
  I want to review the first and second fundamental forms, which I had learned in an undergraduate course but had forgotten. **Lecturer's comment** 第 4 回で mention します. I'll review them in the 4th lecture.
- This is an interesting and useful course. There is no other request. **Lecturer's comment** Thanks.

Q and A

Q 1: Parametrization の例で, Example 1.4 は, 経度が 180° 線上 (両端を含む) を表せていないと思いますが, このように全体を覆っていなくても “parametrization of unit sphere” と表現してもよいのでしょうか. それともより正確には球面からその部分を覗いた曲面の parametrization という意味でしょうか？
Example 1.4 does not represent a longitude on the 180° line (including both ends). I don't think it represents a “parametrization of unit sphere”, as it doesn't cover the whole area. Can we say “parametrization of unit sphere” even if it doesn't cover the whole sphere? Or more precisely, do you mean the “parametrization of the surface of the unit sphere”?

A: そういう意味です. 実は球面全体を覆う parametrization は存在しません.
I mean so. In fact, there is no parametrization which cover the sphere once.

Q 2: 地図の parametrization の 1 つ目の例で, 領域が $[-\pi, \pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ でないのは f を埋め込みにする為でしょうか.
In the first example of parametrization of the sphere, is the reason why the area is not $[-\pi, \pi) \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ to make f embedded?

A: はい. ご質問の集合は $\mathbb{R}^2$ の「領域」ではありませんね.
Yes. The set you are asking about is not a “region” of $\mathbb{R}^2$.

Q 3: Pseudosphere のように特異点があるとき, その存在を図示することなくわかる方法はありますか.
When there is a singularity, such as in the pseudosphere, is there any way to know existence of singularity without illustrating it?

A: はい, 第 4 回以降に説明します.
Yes, we will explain it in the 4th and subsequent lectures.

Q 4: Can the fundamental theorem of surface theory be applied to surfaces, such as a pseudosphere? If so, which parameters in the fundamental forms reflect the smoothness of the surface, or is it to complicated to be represented?

A: The fundamental theorem can be applied for a kind of class of surfaces having singularities, with a slight modification. I'll mention it on the class of 2Q (Adv. Topics in Geom. B1).

Q 5: The fundamental theorem for surface theory seems to be a statement about two dimensional manifolds in $\mathbb{R}^3$, so about submanifolds of codimension $3 - 2 = 1$. Can the theorem be generalized to arbitrary n and not just $n = 3$ for submanifolds of $\mathbb{R}^n$ of dimension $n - 1$?

A: Yes.

Q 6: My question is about the differential of the differential 2-form and its example of exterior differential. I apologize if the question is basic, but I have never encountered the exterior product boffo. After some research, I still wonder how does the exterior product $\wedge$ apply to dx and dy in order to obtain $d(adx + bdy) = (b_x a_y)dx \wedge dy$? (Note: shown ans original. the correct formula is $d(a\,dx + b\,dy) = (b_x - a_y)dx \wedge dy$.)

A: OK, I'll explain it in the lecture (informally).

Q 7: On the blackboard file 3, the lemma states that $d^2 f = 0$ for all functions and used schwarz theorem for the proof. Don't we need to assume that $f \in C^2$ at least?

A: Of course f must be C^2. Sorry that I might drop the declaration: In our context, we assume all quantities are of class C^∞ whenever otherwise stated.

Q 8: Commutativity in terms of differential forms で, f, a, b, c を C^∞ としていますが, f が C^1 のときも $df := f_x\,dx + f_y\,dy$ という定義をしてよいでしょうか.
In the subsection “Commutativity in terms of differential forms”, f, a, b and c are assumed to be C^∞. Can one define $df := f_x\,dx + f_y\,dy$ even if f is of class C^1?

A: はい, よいです. 面倒をさけるため C^∞ にしています（たとえば C^1 だと $d(df)$ の存在が保証されません.）
Yes. We will assume all quantities are of class C^∞ to avoid complexity. For example, if f is of class C^1, the existence of $d(df)$ is not guaranteed.

Q 9: Actually I don't understand first and second fundamental forms *I* and *II*. I think I didn't learn these two concepts during my undergraduate studies. Would you please give me some suggestions about reference materials?

A: Don't worry. These concepts will actually be introduced in 4th lecture. At this time, think that they are only “words”.