Advanced Topics in Geometry A1 (MTH.B405)

Integrability Conditions

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Problem 2-1

Problem (Ex. 2-1)

Find the maximal solution of the initial value problem

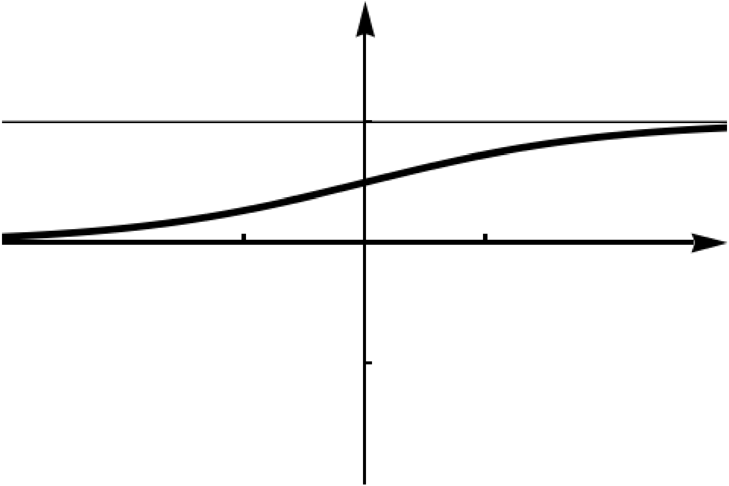
$$\frac{dx}{dt} = x(1-x), \qquad x(0) = a,$$

where a is a real number.

The logistic equation

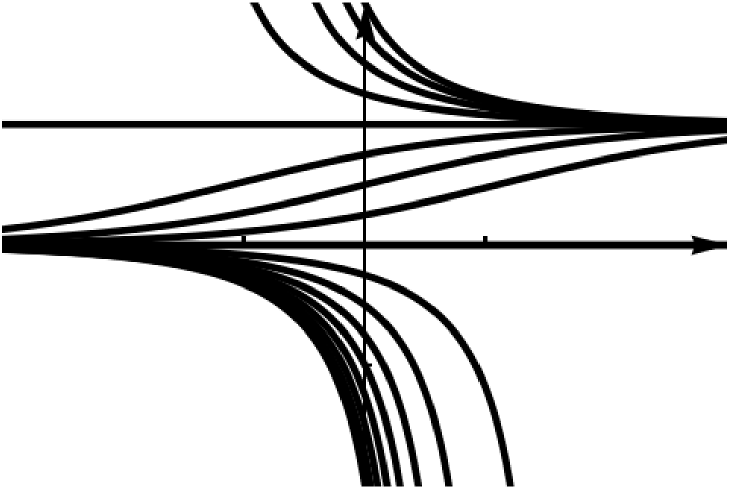
$$x' = x(1-x), \qquad x(0) = a \qquad \text{The Solution:} \quad x(t) = \frac{1}{1 + \frac{1-a}{a} e^{-t}}$$

The logistic curve



$$x(t) = \frac{1}{1 + \frac{1-a}{a} e^{-t}} \ (a \in (0,1))$$

The logistic curve



Exercise 2-2

Problem (Ex. 2-2)

Let $x = x(t)$ be the maximal solution of an initial value problem of differential equation

$$\frac{d^2x}{dt^2} = -\sin x, \qquad x(0) = 0, \quad \frac{dx}{dt}(0) = 2.$$

- *Show that* $\frac{dx}{dt} = 2\cos\frac{x}{2}$.
- *Verify that* x *is defined on* $\mathbb{R}$, *and compute* $\lim_{t\to\pm\infty} x(t)$.

The equation of motion of pendulums

$$\frac{d^2x}{dt^2} + \sin x = 0 \qquad \Rightarrow \qquad \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x = \text{const.}$$

The equation of motion of pendulums: A special solution

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \cos x = \text{const.}$$

$$= \frac{1}{2}\left(\frac{dx}{dt}(0)\right)^2 - \cos x(0) = \frac{1}{2} \times 2^2 - 1 = 1$$

$$(\because \quad x(0) = 0, \quad \dot{x}(0) = 2.)$$

$\Rightarrow$

$$\left(\frac{dx}{dt}\right)^2 = 2(1 + \cos x) = 4\cos^2\frac{x}{2}.$$

$$\boxed{\frac{dx}{dt} = 2\cos\frac{x}{2}}$$

The equation of motion of pendulums: A special solution

$$\frac{dx}{dt} = 2\cos\frac{x}{2} \qquad \Rightarrow \qquad 1 = \frac{1}{2}\sec\frac{x}{2}\frac{dx}{dt}$$

$$t = \int_0^t 1\,dt = \frac{1}{2}\int_0^t \sec\frac{x(t)}{2}\frac{dx(t)}{dt}\,dt = \frac{1}{2}\int_{x(0)}^{x(t)} \sec\frac{x}{2}\,dx$$

$$\boxed{x(t) = 4\tan^{-1}\tanh\frac{t}{2}}$$

The equation of motion of pendulums: A special solution

$$\boxed{x(t) = 4\tan^{-1}\tanh\frac{t}{2}}$$

Exercise 2-3

Problem (Ex. 2-3)

Find an explicit expression of a space curve $\gamma(s)$ *parametrized by the arc-length* s*, whose curvature* κ *and torsion* τ *satisfy*

$$\kappa = \tau = \frac{1}{\sqrt{2}(1+s^2)}.$$

Strategy: Solve

$$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, \qquad \mathcal{F}(0) = \mathrm{id}$$

for $\mathcal{F}(s) = (\boldsymbol{e}(s), \boldsymbol{n}(s), \boldsymbol{b}(s))$.
Then the desired curve is obtained by

$$\gamma(s) = \int_0^s \boldsymbol{e}(s)\, ds$$

Frenet-Serret equation

$$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} = \frac{1}{1+s^2}\mathcal{F}\Omega, \qquad \Omega := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Change Variable $\boxed{s = \tan t}$ $\Rightarrow$

$$\frac{d}{dt}\mathcal{F} = \mathcal{F}\Omega$$

With initial condition $\mathcal{F}(0) = \mathrm{id}$,

$$\mathcal{F}(t) = \exp t\Omega = \mathrm{id} + \sum_{k=1}^{\infty} \frac{t^k}{k!}\Omega^k$$

Frenet-Serret equation

$$\mathcal{F}(t) = \exp t\Omega = \mathrm{id} + \sum_{k=1}^{\infty} \frac{t^k}{k!}\Omega^k \qquad \Omega := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Note:

$$\Omega^2 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \Omega^3 = -\Omega, \qquad \Omega^4 = -\Omega^2.$$

$\Rightarrow$

$$\exp t\Omega = \begin{pmatrix} \frac{1}{2}(1+\cos t) & * & * \\ \frac{1}{\sqrt{2}}\sin t & * & * \\ \frac{1}{2}(1-\cos t) & * & * \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1+\frac{1}{\sqrt{1+s^2}}\right) & * & * \\ \frac{1}{\sqrt{2}}\frac{s}{\sqrt{1+s^2}} & * & * \\ \frac{1}{\sqrt{2}}\left(1-\frac{1}{\sqrt{1+s^2}}\right) & * & * \end{pmatrix}$$

Frenet-Serret equation

$$\boldsymbol{e}(s) = \gamma'(s) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{1+s^2}}\right) \\ \frac{1}{\sqrt{2}}\frac{s}{\sqrt{1+s^2}} \\ \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{1+s^2}}\right) \end{pmatrix}$$

$$\gamma(s) = \begin{pmatrix} \frac{1}{2}\left(s + \log(s + \sqrt{1+s^2})\right) \\ \frac{1}{\sqrt{2}}\sqrt{1+s^2} \\ \frac{1}{2}\left(s - \log(s + \sqrt{1+s^2})\right) \end{pmatrix}$$

Frenet-Serret equation: an alternative solution

$$\frac{d}{ds}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix} \qquad \because \quad \kappa = \tau$$

Set

$$\mathcal{G} := \mathcal{F}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

then

$$\frac{d}{ds}\mathcal{G} = \mathcal{G}\Lambda, \qquad \Lambda := \frac{1}{1+s^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$