# Advanced Topics in Geometry A1 (MTH.B405)

Integrability Conditions

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# Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{*}$$

**Proposition (Prop. 3.2)**

*If a matrix-valued $C^\infty$ function $X\colon U \to \mathrm{GL}(n, \mathbb{R})$ satisfies $(*)$, it holds that*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j$$

*for each $(j, k)$ with $1 \leqq j < k \leqq m$.*

# Integrability of Linear systems

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \tag{1}$$

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j \tag{2}$$

**Theorem (Thm. 3.5)**

*Let* $\Omega_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* $C^\infty$*-functions defined on a simply connected domain* $U \subset \mathbb{R}^m$ *satisfying* $(2)$*. Then for each* $\mathrm{P}_0 \in U$ *and* $X_0 \in \mathrm{M}_n(\mathbb{R})$*, there exists the unique* $n \times n$*-matrix valued function* $X \colon U \to \mathrm{M}_n(\mathbb{R})$ *satisfying* $(1)$

# Integrability Conditions

## Lemma (Lem. 3.4)

*Let* $\Omega_j \colon U \to \mathrm{M}_n(\mathbb{R})$ $(j = 1, \dots, m)$ *be* $C^\infty$*-maps defined on a domain* $U \subset \mathbb{R}^m$ *which satisfy*

$$\frac{\partial \Omega_j}{\partial u^k} - \frac{\partial \Omega_k}{\partial u^j} = \Omega_j \Omega_k - \Omega_k \Omega_j.$$

*Then for each smooth map*

$$\sigma \colon D \ni (t, w) \longmapsto \sigma(t, w) = (u^1(t, w), \dots, u^m(t, w)) \in U$$

*defined on a domain* $D \subset \mathbb{R}^2$*, it holds that*

$$\frac{\partial T}{\partial w} - \frac{\partial W}{\partial t} - TW + WT = 0,$$

*where* $T := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial t}$, $W := \sum_{j=1}^m \widetilde{\Omega}_j \frac{\partial u^j}{\partial w}$, $(\widetilde{\Omega}_j := \Omega_j \circ \sigma)$.

# Integrability Conditions

$$\frac{\partial X}{\partial u^j} = X\Omega_j \quad (j = 1, \dots, m), \qquad X(\mathrm{P}_0) = X_0. \qquad (*)$$

## Lemma (Lem. 3.3)

*Let $X\colon U \to \mathrm{M}_n(\mathbb{R})$ be a $C^\infty$-map satisfying $(*)$ Then for each smooth path $\gamma\colon I \to U$ defined on an interval $I \subset \mathbb{R}$, $\hat{X} := X \circ \gamma : I \to \mathrm{M}_n(\mathbb{R})$ satisfies the ordinary differential equation*

$$\frac{d\hat{X}}{dt}(t) = \hat{X}(t)\Omega_\gamma(t) \quad \left(\Omega_\gamma(t) := \sum_{j=1}^{n} \Omega_j \circ \gamma(t) \frac{du^j}{dt}(t)\right)$$

*on $I$, where $\gamma(t) = \big(u^1(t), \dots, u^m(t)\big)$.*

# Proof of Theorem 2.5 (outline)

- Take $\mathrm{P} \in U$, and a path $\gamma \colon [0, 1] \to U$ with $\gamma(0) = \mathrm{P}_0$ and $\gamma(1) = \mathrm{P}$.
- Solve the linear ODE as in Lemma 2.3 with initial condition $\hat{X}(0) = X_0$.
- Show the value $\hat{X}(1)$ does not depend on $\gamma$ $\Leftarrow$ by Lem. 3.4
- Define $X(\mathrm{P}) := \hat{X}(1)$.
- Show $X$ is the desired solution.

# Application: Poincaré's lemma

**Theorem (Poincaré's lemma)**

*If a differential* $1$*-form*

$$\omega = \sum_{j=1}^{m} \alpha_j(u^1, \ldots, u^m)\, du^j$$

*defined on a simply connected domain* $U \subset \mathbb{R}^m$ *is closed, that is,* $d\omega = 0$ *holds, then there exists a* $C^\infty$*-function* $f$ *on* $U$ *such that* $df = \omega$*. Such a function* $f$ *is unique up to additive constants.*

# Application: Conjugation of harmonic functions

**Theorem**

*Let* $U \subset \mathbb{C} = \mathbb{R}^2$ *be a simply connected domain and* $\xi(u, v)$ *a* $C^\infty$*-function harmonic on* $U$*. Then there exists a* $C^\infty$ *harmonic function* $\eta$ *on* $U$ *such that* $\xi(u, v) + \mathrm{i}\,\eta(u, v)$ *is holomorphic on* $U$*.*

# Application: Conjugation of harmonic functions

## Example

$\xi(u, v) = e^u \cos v$

# Exercise 3-1

## Problem

*Let*

$$\xi_1(u,v) := \frac{u}{u^2+v^2}, \qquad \xi_2(u,v) := \log\sqrt{u^2+v^2}$$

*be a function defined on non-simply connected domain* $U := \mathbb{R}^2 \setminus \{(0,0)\}$*.*

1. *Show that both* $\xi_1$ *and* $\xi_2$ *are harmonic on* $U$*.*
2. *Verify that there exists a conjugate harmonic functaion* $\eta_1$ *of* $\xi_1$ *on* $U$*.*
3. *Prove that there exists no conjugate harmonic functaion* $\eta_2$ *of* $\xi_2$ *on* $U$*.*

# Exercise 3-2

## Problem

*Consider a linear system of partial differential equations for $3 \times 3$-matrix valued unknown $X$ on a domain $U \subset \mathbb{R}^2$ as*

$$\frac{\partial X}{\partial u} = X\Omega, \quad \frac{\partial X}{\partial v} = X\Lambda, \\ \left(\Omega := \begin{pmatrix} 0 & -\alpha & -h_1^1 \\ \alpha & 0 & -h_1^2 \\ h_1^1 & h_1^2 & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} 0 & -\beta & -h_2^1 \\ \beta & 0 & -h_2^2 \\ h_2^1 & h_2^2 & 0 \end{pmatrix}\right),$$

*where $(u, v)$ are the canonical coordinate system of $\mathbb{R}^2$, and $\alpha$, $\beta$ and $h_j^i$ $(i, j = 1, 2)$ are smooth functions defined on $U$. Write down the integrability conditions in terms of $\alpha$, $\beta$ and $h_j^i$.*