Advanced Topics in Geometry A1 (MTH.B405)

The fundamental theorem for surfaces

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$\boxed{6-3=3}$

Given data: six functions defined on $U \subset \mathbb{R}^2$.

given.

$$\widehat{I} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad \widehat{II} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix},$$

$(g_{ij} = P_{,i} \cdot P_{,j})$

Assumption:

$$g_{11} > 0, \qquad g_{22} > 0, \quad \text{and} \qquad g_{11}g_{22} - g_{12}g_{21} > 0$$

$(|P_{,1} \times P_{,2}|^2)$

Set up:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}(g_{lj,k} + g_{il,j} - g_{jl,k}), \qquad A^i_j = \sum_{l=1}^{2} g_{jl}h_{il}$$

Christoffel symbols

Weingarten matrix

- Gauss & Codazzi equations: written in terms of g & h.

The statement

Theorem (Theorem 6.1)

domain in $\mathbb{R}^2$

Assume U is simply connected, and (g_{ij}) and (h_{ij}) satisfy the Gauss equation and Codazzi equations. Then there exists a regular surface $p\colon U \to \mathbb{R}^3$ such that

existence

- *the first fundamental form of p is $ds^2 = \sum_{i,j} g_{ij} du^i\, du^j$,*
- *the second fundamental form of p with respect to the unit normal vector field $\nu := (p_{,1} \times p_{,2})/|p_{,1} \times p_{,2}|$ coincides with $II = \sum_{i,j} h_{ij} du^i\, du^j$.*

Moreover, such a surface p is unique up to a transformation congruence.

uniqueness

$$p \mapsto Rp + \boldsymbol{a}, \qquad R \in \mathrm{SO}(3),\ \boldsymbol{a} \in \mathbb{R}^3.$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

congruent transformation

Uniqueness

$p, \hat{p}$: surfaces with common (g_{ij}) (h_{ij})

- $\mathcal{F}, \hat{\mathcal{F}}$: Gauss frames of $p, \hat{p}$, resp.
- $\dfrac{\partial \mathcal{F}}{\partial u^i} = \mathcal{F}\Omega_i$ $\quad \dfrac{\partial \hat{\mathcal{F}}}{\partial u^i} = \hat{\mathcal{F}}\Omega_i$ $\quad$ determind by (g_{ij}) (h_{ij})
- $\hat{\mathcal{F}} = R\mathcal{F}$ $\quad$ R: constant regular matrix

  $\left(\because \dfrac{\partial}{\partial u^i}(\hat{\mathcal{F}}\mathcal{F}^{-1}) \cdots = 0\right)$
- $\mathcal{F}^T\mathcal{F} = \begin{pmatrix} p_{,1}\cdot p_{,1} & p_2\cdot p_1 & 0 \\ p_{,1}\cdot p_2 & p_2\cdot p_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

  $= \hat{\mathcal{F}}^T\hat{\mathcal{F}}$ $\quad R^T R = \mathrm{id}$

  R = orthogonal. $\quad \hat{p} = Rp + a \quad \left(\begin{array}{l} R \in SO(3) \\ a \in \mathbb{R}^3 \end{array}\right.$

$dp = R\,d\hat{p}$ $\quad \det R = 1$ $\quad \boxed{R \in SO(3)}$

Existence

appropriate initial cond

g_{ij} h_{ij}

Solve $\dfrac{\partial \mathcal{F}}{\partial u^j} = \mathcal{F}\Omega_j$ with $\mathcal{F}(u_0, v_0) =$

Gauss - Codazzi eq.

$\mathcal{F} = (\mathbf{a}_1 \quad \mathbf{a}_2 \quad \nu)$

- $d\mathbf{p} = \mathbf{a}_1 du + \mathbf{a}_2 dv \quad \leftarrow (\mathbf{p}_u du + \mathbf{p}_v dv)$

$d(d\mathbf{p}) = 0$

$\downarrow$ Poincaré

$\exists \mathbf{p}(u, v)$ • Show $\mathbf{p}$ is the disired one

Exercise 6-1

additonal condition.

$\boxed{0<\theta<\pi}$

Problem (Ex. 6-1)

Let $\theta\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming θ satisfies $\boxed{\theta_{uv} = \sin\theta,}$ prove that there exists a surface $p\colon U \to \mathbb{R}^3$ whose first and second fundamental forms are

cf. Ex 5-1

- $ds^2 = du^2 + 2\cos\theta\,du\,dv + dv^2, \qquad II = 2\sin\theta\,du\,dv.$

Gauss. eq $\Leftrightarrow \theta_{uv} = \sin\theta$

Exercise 6-2

Ex 5-2

Problem (Ex. 6-2)

Let $\sigma\colon U \to \mathbb{R}$ be a C^∞-function defined on a simply connected domain U of the uv-plane $\mathbb{R}^2$. Assuming σ satisfies $\boxed{\Delta\sigma = -\frac{1}{2}\sinh\sigma}$, prove that there exists a surface $p\colon U \to \mathbb{R}^3$ with

Gauss eq.

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad II = \frac{1}{2}\big(\underbrace{(e^{2\sigma}+1)}_{L}du^2 + \underbrace{0}_{}\, + \underbrace{(e^{2\sigma}-1)}_{N}dv^2\big).$$

$f(z) = \dfrac{L-N}{2} - iM = 1$; ($H = \dfrac{1}{2}e^{-2\sigma}(L+N)$) $= \dfrac{1}{2}$ const

"codazzi" satisfied.

☆ Counterexample theorem of Hopf's conjecture.

Hopf's problem

Are there closed surfaces of constant mean curvature different from the round sphere?

$\Rightarrow$ Yes $\exists$ constant mean curvature Tori

H. Wente 1980's