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Info. Sheet 6; Advanced Topics in Geometry A1 (MTH.B405)

Corrections

- Handout C/Blackboard C, page 4, the 2nd line of the formula of R_{jk}:
$$-\sum_{i,s} g_{is}(\Gamma^s_{ks}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j}) \quad\Rightarrow\quad -\sum_{i,s} g_{is}(\Gamma^s_{k2}\Gamma^i_{1j} - \Gamma^s_{k1}\Gamma^i_{2j})$$
- Handout C/Blackboard C, page 6, title ingetrablility ⇒ integrability
- Lecture Note, page 28, 6th line of the equation "I^j_3-":
$$-\sum_l g^{jl}(h_{1l,2}-h_{l2,1})+\sum_{l,\alpha,\beta} g^{\alpha j}g^{l\beta}\sum_s \Big((g_{\alpha s}\Gamma^s_{\beta 2} + g_{s\beta}\Gamma^s_{2\beta})h_{1l} - (g_{\alpha s}\Gamma^s_{\beta 1} + g_{s\beta}\Gamma^s_{1\beta})h_{l2}\Big)$$
$$\Rightarrow\quad -\sum_l g^{jl}(h_{1l,2}-h_{l2,1})+\sum_{l,\alpha,\beta} g^{\alpha j}g^{l\beta}\sum_s \Big((g_{\alpha s}\Gamma^s_{\beta 2} + g_{s\beta}\Gamma^s_{2\alpha})h_{1l} - (g_{\alpha s}\Gamma^s_{\beta 1} + g_{s\beta}\Gamma^s_{1\alpha})h_{l2}\Big)$$
- Lecture Note, page 29, line 7:
$$\sum_i g_{ik}I^i_j = (h_{l1}h_{j2}-h_{l2}h_{j1}) = R_{jk}+h_{k1}h_{j2}-h_{k2}h_{j1} \quad\Rightarrow\quad \sum_i g_{ik}I^i_j = R_{jk}+h_{k1}h_{j2}-h_{k2}h_{j1}$$
- Ex 5-2 の問題文に σ: smooth の仮定が必要だと思います．
I think the assumption of the smoothness of σ is necessary in the problem 5-2.

Students' comments

- 計算がそれなりに大変に感じますが，微分幾何プロにとってはやはり朝飯前なんですか？
I feel that the calculations are quite hard, but is it a piece of cake for a differential geometry professional?

  Lecturer's comment No, it's not.

- 第 2 回の Lecture Note の修正版，ありますか？
Is there a revised version of the Lecture Note 2?

  Lecturer's comment I'm sorry. I've uploaded it.

- Ex 5-2 の問題文に σ: smooth の仮定が必要だと思います．
I think the assumption of the smoothness of σ is necessary in the problem 5-2.

  Lecturer's comment Yes, see page 5 of the Blackboard C for the lecture of April 11th.

Q and A

Q 1: (問題 5-2 について) 確かに計算したら同じになることは分かりましたが，この $q(z)$ とやらはどこから思いついたのでしょうか．別の計算や幾何学的な意味を背景とする関数なのですか．
For Problem 5-2: I could see that the calculations would show the conclusion, but where did you come up with this $q(z)$ or whatever it is? Is it a function with a different computational or geometric meaning behind it?

A: Using complex coordinate $z = u + iv$, the second fundamental form is written as
$$\begin{aligned} I\!I &= L\,du^2 + 2M\,du\,dv + N\,dv^2 \\ &= \frac{1}{4}\left(L\,(dz+d\bar z)^2 + 2M\,(dz+d\bar z)(dz-d\bar z) - N\,(dz-d\bar z)^2\right) \\ &= \frac{1}{4}((L-N)-2iM)\,dz^2 + (L+N)\,dz\,d\bar z + \frac{1}{4}(L-N)+2iM)\,d\bar z^2 \\ &= \frac{q}{2}\,dz^2 + 2e^{2\sigma}H\,dz\,d\bar z + \frac{\bar q}{2}d\bar z^2, \end{aligned}$$
because of $dz = du + i\,dv$, $d\bar z = u - i\,dv$. That is, $q/2$ is the coefficient of "$(2,0)$-part" of the second fundamental form with respect to the complex coordinate z.

Q 2: Ex 5-2 で H が定数関数だと q が正則となりますが，逆に任意の正則関数 f に対し $q = f$ かつ H が定数関数になるような曲面をとることはできるのでしょうか.
In Ex 5-2, q is regular if H is a constant function, but conversely, can we take a surface such that $q = f$ and H is a constant function for any regular function f?

A: Yes. It seems to be obvious, but it requires some preparation to explain.

Q 3: If the first and second fundamental forms that satisfy G-W formula are given, does there exist a unique surface corresponding to them?

A: Yes, if they satisfy the Gauss and Codazzi equations, not Gauss-Weingarten.

Q 4: If all g_{ij} and K are given, can we deduce all h_{ij}?

A: No. A counter example: Let

$$p_1(u, v) = (u, v, 0), \qquad p_2(u, v) = (\cos u, \sin u, v)$$

share common first fundamental form $ds^2 = du^2 + dv^2$ and Gaussian curvature $K = 0$.

Q 5: (5.5) の表示はどのような場面で使われますか？
In what situations is the expression (5.5) used?

A: I will not use it this time. It will be used in the sense of sectional curvature when one generalize the theory for higher dimensions.

Q 6: 問題 5-1 で E, G がそれぞれ u, v の関数になっていることは分かりましたが，問題の仮定と合わせても幾何学的なイメージが分かりません．問題 5-1 はかなり強そうな条件ですが，これも幾何学的な性質についてあまり言うことはできないのでしょうか？
I could see in problem 5-1 that E and G are functions of u and v, respectively, but the geometrical meaning is not clear, even with the assumptions of the problem. The assumption of problem 5-1 seems to be a very strong, but not much can be said about its geometrical properties, can it? Can you say much about the geometrical properties?

A: The result in Problem 5-1 is related to the existence of “asymptotic Chebyshev net” for pseudospherical surfaces. This fact is one of the important tool in the lectures in 2Q, where I'll try to figure out the geometric meaning of it.

Q 7: Codazzi equations が 3 つの式のうち 2 つのみを指しますが，この 2 つだけでも有用なのでしょうか.
Codazzi equations refers to only two of the three equations. Are these two alone useful?

A: Yes. For example, Problem 5-1 uses only Codazzi equations. On the other hand, the Gauss equation itself has a remarkable meaning. Search “Theorema Egregium of Gauss”.

Q 8: On the blackboard, it says that we can assume that $K = -1$ and indeed if we assume this, then the equation can be simplified nicely. But what about the case $K \neq -1$? Also I do not understand how to conclude that $E_u = 0$, $G_v = 0$ from the above equations, unless $F = 0$.

A: The situation is quite same even when $K \neq -1$.